Overdue charges will be levied under rules for each day if the book is kept beyond the date stamed above.

2-8 → 2

| | |
|---|---|
| ASL-259 | Intermediate Coordinate Geometry<br>Prem Singh, V Parkash, Jan Mohammad,<br>Amrik Singh -<br>Gian Chand & Co Ambala Cantt. |
| (211 Pages)<br>ASL-260 | Enduring Passion<br>Dr. Marie Stopes ( 1953 ) 163 Pages |
| ASL-261 | Wonders of the Past & Present<br>C.A. Parkhurst - 1949<br>Macmillan & Company Ltd. London<br>(151 Pages) |

| | |
|---|---|
| ASL-259 | Intermediate Coordinate Geometry |
| | Prem Singh, V Prakash, Jan Mohammed, |
| | Amrik Singh - |
| (211 Pages) | Gian Chand & Co Ambala Cantt. |
| ASL-260 | Enduring Passion |
| | Dr. Marie Stopes (1953) 163 pgs |
| ASL-261 | Wonders of the Past & Present |
| | C.A. Parkhurst. 1949 |
| | Macmillan & company Ltd. London |
| | (151 Pages) |

INTERMEDIATE

# CO-ORDINATE GEOMETRY


*by*

PREM SINGH, M.A.

*Professor of Mathematics*
*Arya College, Ludhiana.*

V. PARKASH, M.A.

*Head of the Mathematics Deptt.*
*D. S. College, Karnal.*

JAN MOHAMMAD, M.A.

*Professor of Mathematics,*
*Amar Singh College, Sri Nagar.*

*Revised by*

AMRIK SINGH, M.A.

*Member of the Board of studies in Mathematics, Punjab University*
*and Head of Mathematics Deptt. Khalsa College, Amritsar.*



AMBALA CANTT

GIAN CHAND & Co.

EDUCATIONAL PUBLISHERS

*Rs. 4/-*

PRINTED AT THE OXFORD AND CAMBRIDGE PRESS, DELHI AND PUBLISHED BY GIAN GHAND SHARMA, FOR GIAN CHAND AND CO., AMBALA CANTT.

# PREFACE

This book is intended to present a complete course of Intermediate Co-ordinate Geometry according to the revised syllabus of the Punjab University for Intermediate Classes in Co-ordinate Geometry.

The authors do not claim any originality about the subject matter but have tried to put the subject in a most presentable form in the light of their long teaching experience.

The book has been thoroughly revised. Answers have been checked. A set of Miscellaneous examples has been added at the end of the book for good students. The authors hope that the book will meet the need of all types of students particularly the average student. A new chapter on Hyperbola has been added at the end of the book.

The authors acknowledge their indebtedness to several standard books on Co-ordinate Geometry which were freely consulted during the preparation of the book. Suggestions and criticisms for improvement of the book from fellow teachers will be most cordially received.

Ambala Cantt.

AUTHORS

# PREFACE TO THE THIRD EDITION

The whole book has been thoroughly revised for this edition. The chapters third to sixth have been entirely re-written. It is hoped that in its present form the book will prove more useful.

The present edition has been printed very carefully and it is hoped that there will be very few errors.

AUTHORS

# CONTENTS

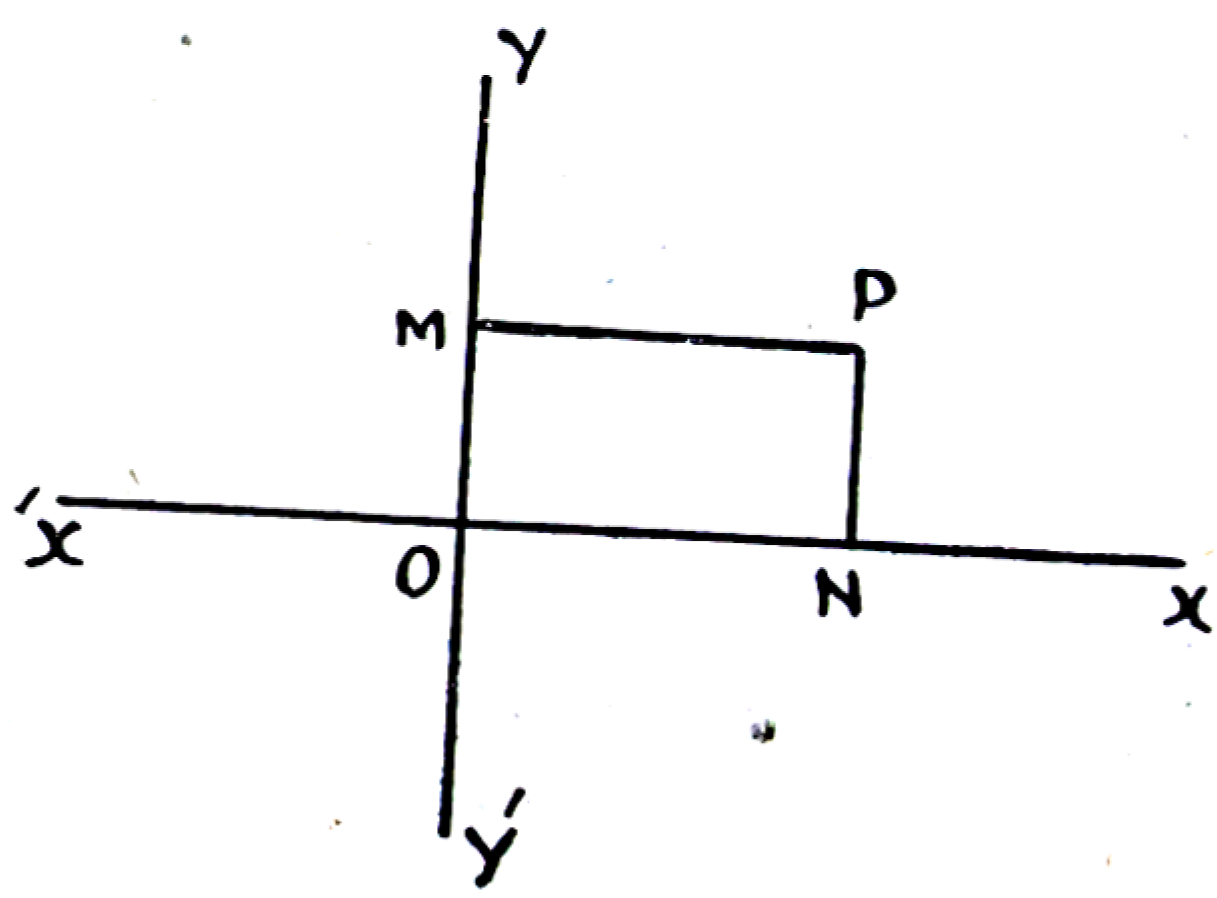
CHAPTER I

# CO-ORDINATES

**1.1. Co-ordinate Geometry** is a method of proving geometrical theorems by algebraical processes. If we limit our considerations to the study of figures in a plane, the position of a point is determined by its distances from two fixed lines in the plane. These distances can be measured numerically in terms of a given unit or may be denoted by algebraical symbols called the co-ordinates of the point.

**1.2. Cartesian Co-ordinates.** Let $X'OX$ and $Y'OY$ be two fixed straight lines at right angles to one another. $X'OX$ is drawn from left to right and $Y'OY$ upwards. These are called **Co-ordinate axes.**

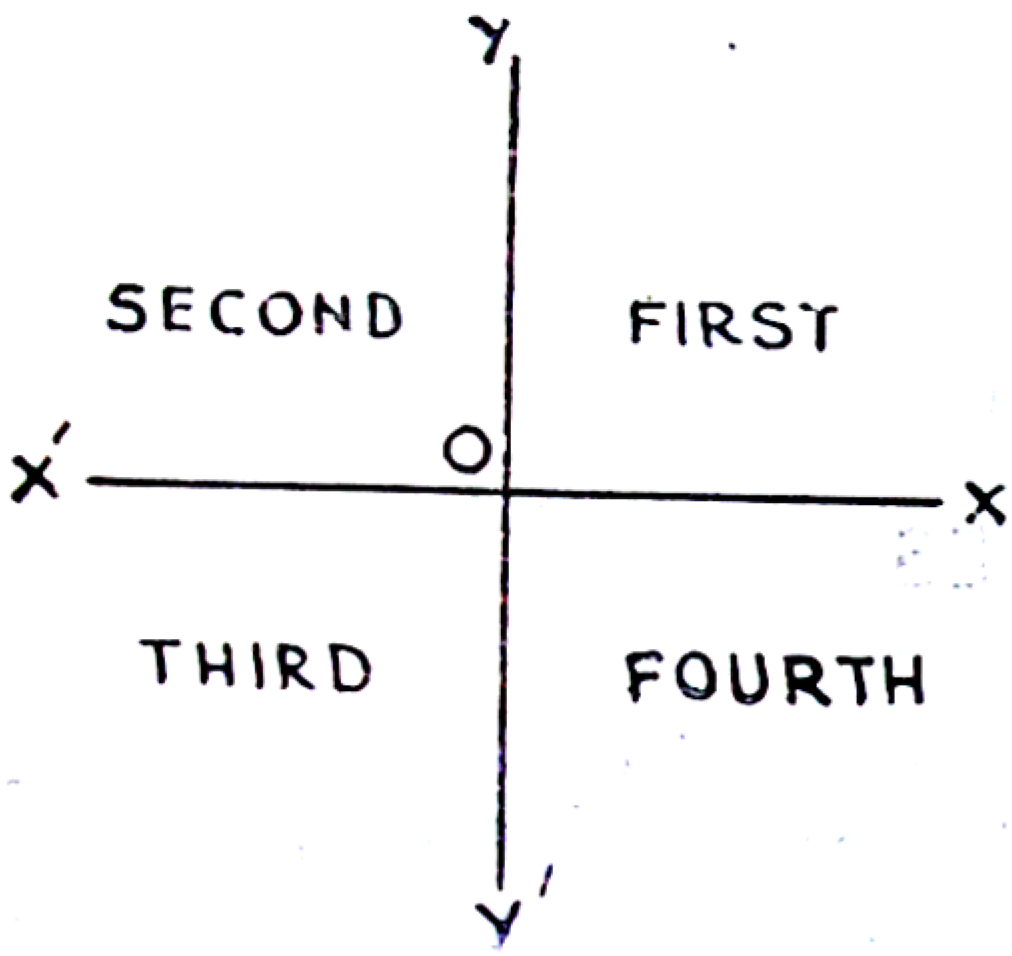
The point O is called the **origin of co-ordinates** or simply the **origin.**




The position of any point P in the plane of paper is determined by the distances MP and NP from the axes. These distances are called the **co-ordinates** of the point P.

ON the distance of P, from OY parallel to OX is called the **abscissa** or $x$ **co-ordinate** of P and is denoted by $x$. NP the distance of P, from OX parallel to OY is called the **ordinate** or Y **co-ordinate** of P and is denoted by $y$. If the co-ordinates of a point P are $l$ and $m$ units respectively, then at the point P, $x=l$ and $y=b$. The point P in this case is briefly referred to as the point $(l, m)$, $l$ being the abcissa and $m$ the ordinate of the point. The co-ordinates are called **Cartesian** after the French mathematician **Rene Descartes** (1596—1650) who used them in his treatise on *Geometry* (1637).

**1.21. Signs of Cartesian Co-ordinates.** The straight lines X'OX and Y'OY divide the plane into four parts. Each part is called a quadrant. The quadrants XOY, YOX', X'OY' and Y'OX are respectively called the **first**, **second** **third** and **fourth** quadrants.




Since the distances measured along or parallel to OX and OY are considered **positive** and those measured along or parallel to OX' and OY' are considered *negative* it follows that :

(*i*) for a point in the *first* quadrant both the abscissa and the ordinate are *positive* ; (*ii*) for a point in the *second* quadrant, the abscissa is *negative* and the ordinate is *positive* ;

(*iii*) for a point in the *third* quadrant both the abscissa and the ordinate are negative; and

(*iv*) for a point in the *fourth* quadrant the abscissa is *positive* and the ordinate negative.

It should be noted that the sign of the co-ordinates determines the quadrant in which a point lies. Thus a point $(-2, 3)$ lies in the second quadrant and $(2, -3)$ lies in the fourth.

**1.22. Plotting of Points.** To plot a point whose co-ordinates are $(l, m)$ the following method is followed :

Take a number of units equal to $l$ along OX or OX' according as $l$ is positive or negative and then from the point so determined measure off a number of units equal to $m$ parallel to Y-axis in the direction of OY or OY' according as $m$ is positive or negative.

It should be remembered that the co-ordinates of the origin are $(0, 0)$; the ordinate of any point on $x$-axis is zero; and the abscissa of any point on $y$-axis is zero.

## Exercise I (a)

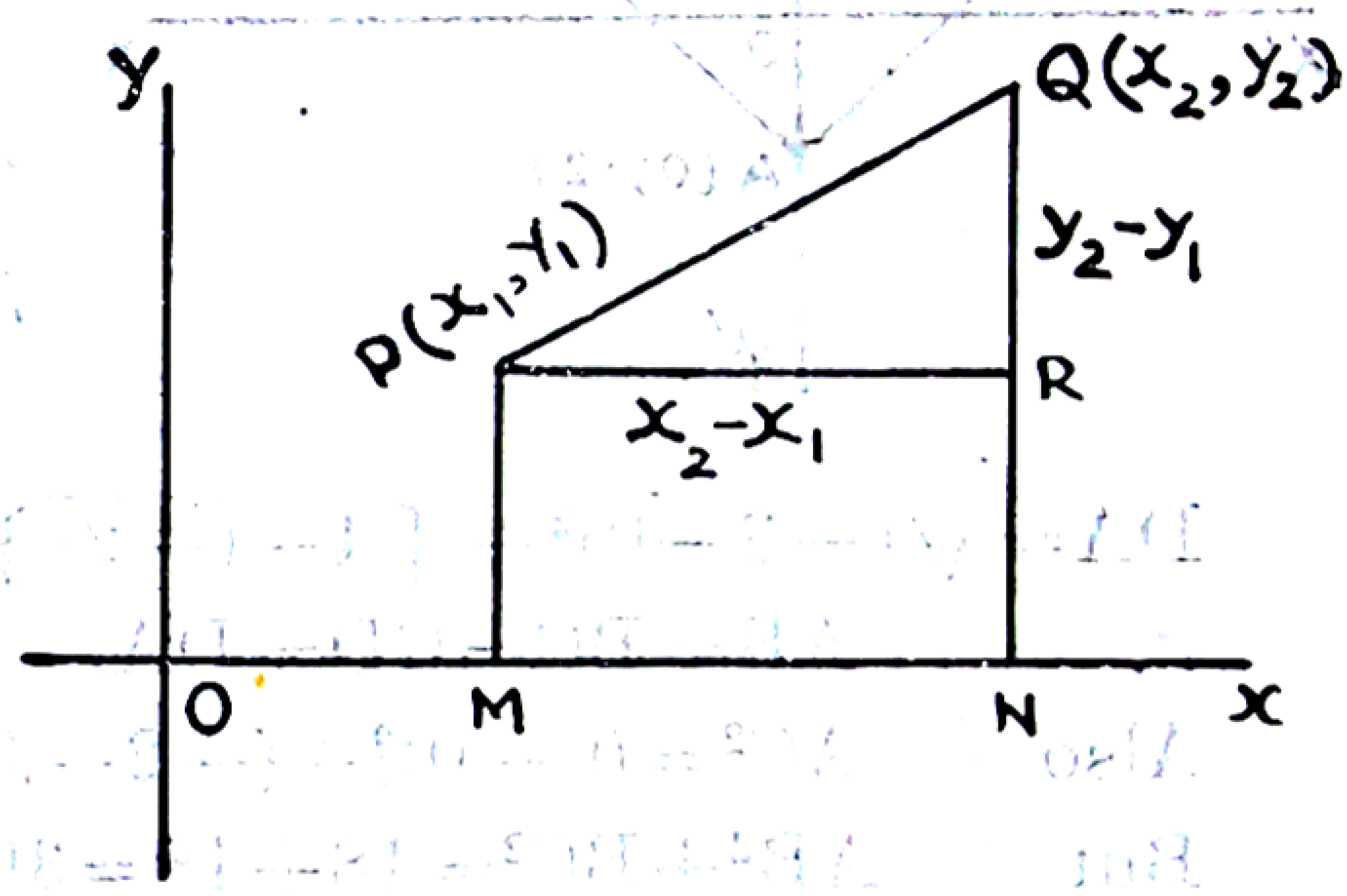
1. Plot the following points :
(3, 7) ; (−4, 5) ; (2, 0) ; (0, −7) ; (−4, −5) ; (−6, 0).

2. Construct the quadrilateral whose vertices are
(2, 5) ; (7, 0) ; (−5, 0) ; (−4, −7)

3. The sides of a right-angled triangle are $a$ and $b$. Taking these sides as the axes of co-ordinates, find the co-ordinates of the vertices.

### 1.3. Distance between two given points.

Let $P(x_1\ y_1)$ and $Q(x_2 y_2)$ be the two given points. Draw PM and QN perpendiculars to OX and PR perpendicular to NQ.




Now $OM=x_1$, $MP=y_1$, $ON=x_2$ and $NQ=y_2$

From the right-angled △ PRQ

we have $PQ^2=PR^2+RQ^2$

But $PR=MN=ON-OM$
$=x_2-x_1$

and $RQ=NQ-NR$
$=y_2-y_1$

$$\therefore \quad PQ^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$$\text{or} \quad PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

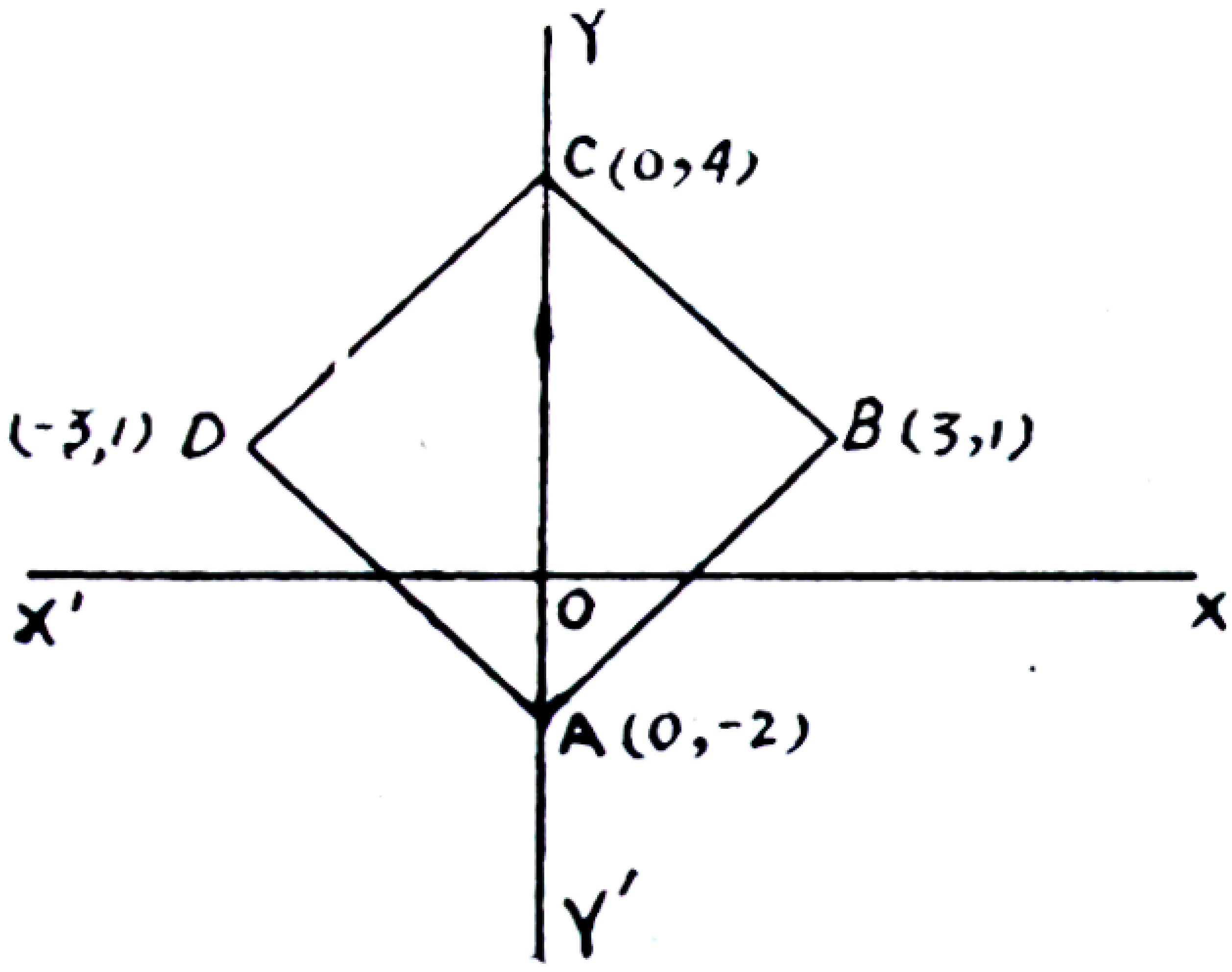
**Cor.** The distance of the point Q $(x_2, y_2)$ from the origin O (0, 0) is obtained by putting $x_1=0$ and $y_1=0$ in the above formula and is $\sqrt{x_2^2+y_2^2}$.

**Note.** It must be remembered that the formulae of co-ordinate geometry are all general. They hold true for points situated in any quadrant provided that the proper signs are always affixed to the numerical values of the co-ordinates when they are introduced.

**Example 1.** *Find the distance between the points* (−1, 3), (−7, −5).

The required distance $=\sqrt{\{-1-(-7)\}^2+\{3-(-5)\}^2}$
$=\sqrt{36+64} = 10.$

**Example 2.** *Show that the points* $(0,-2)$, $(3,1)$, $(0,4)$ *and* $(-3, 1)$ *are the vertices of a square.* (*P.U. 1936*)




Plotting the points roughly we see that the vertices are : $A(0,-2)$; $B(3, 1)$ ; $C(0, 4)$ ; $D(-3, 1)$.

Now

$AB=\sqrt{(0-3)^2+(-2-1)^2}$
$=\sqrt{18}$

$AC=\sqrt{(3-0)^2+(1-4)^2}$
$=\sqrt{18}$

$CD=\sqrt{\{0-(-3)\}^2+(4-1)^2}=\sqrt{18}$

$DA=\sqrt{(-3-0)^2+\{1-(-2)\}^2}=\sqrt{18}$

$\therefore \qquad AB=BC=CD=DA.$

Also $\qquad AC^2=(0-0)^2+(-2-4)^2=36.$

But $\qquad AB^2+BC^2=18+18=36.$

$\therefore \qquad AC^2=AB^2+BC^2$

$\therefore \qquad \angle ABC=90$

Hence $\qquad$ ABCD is a square.

## Exercise I (b)

1. Find the distance between the following pairs of points :

(*i*) $(-2, 1)$ and $(-6, -2)$ (*ii*) $(1, -1)$ and $(-2, 1)$

(*iii*) $(3, 4)$ and $(2, -2)$ (*iv*) $(a, b)$ and $(-b, a)$

(*v*) $(a+b)$, $(c+a)$ and $(c+a, b+c)$

(*vi*) $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, (a \sin \beta,)$

2. Show that the points $(3, 0)$, $(6, 4)$ and $(-1, 3)$ are the vertices of a right-angled triangle.

3. Show that the point $(2, -1)$ is equidistant from the three points $(2, 4)$, $(5, 3)$ and $6, 2)$.

4. Show that the four points (5, 2) (3, 7), (—1, 4) and (1,—1) taken in order are the angular points of a parallelogram.

5. Prove that the points (—1, 0), (0, 3), (3, 2) and (2, —1) are at the corners of a square.

6 Prove that the points (0, —1); (2, 1); (0, 3) and (—2, 1) are the vertices of a square. *(P. U. 1944)*

7. Show that the points (—3, 4), (—2, 0), (1, 5) are equidistant from the point $(-\frac{1}{2}, \frac{5}{2})$.

8. If the point $(x, y)$ is equidistant from the points (2, 3) and (3, 2) prove that $x=y$.

9. If the point $(x, y)$ is at a distance 5 from the point (2, 3) show that

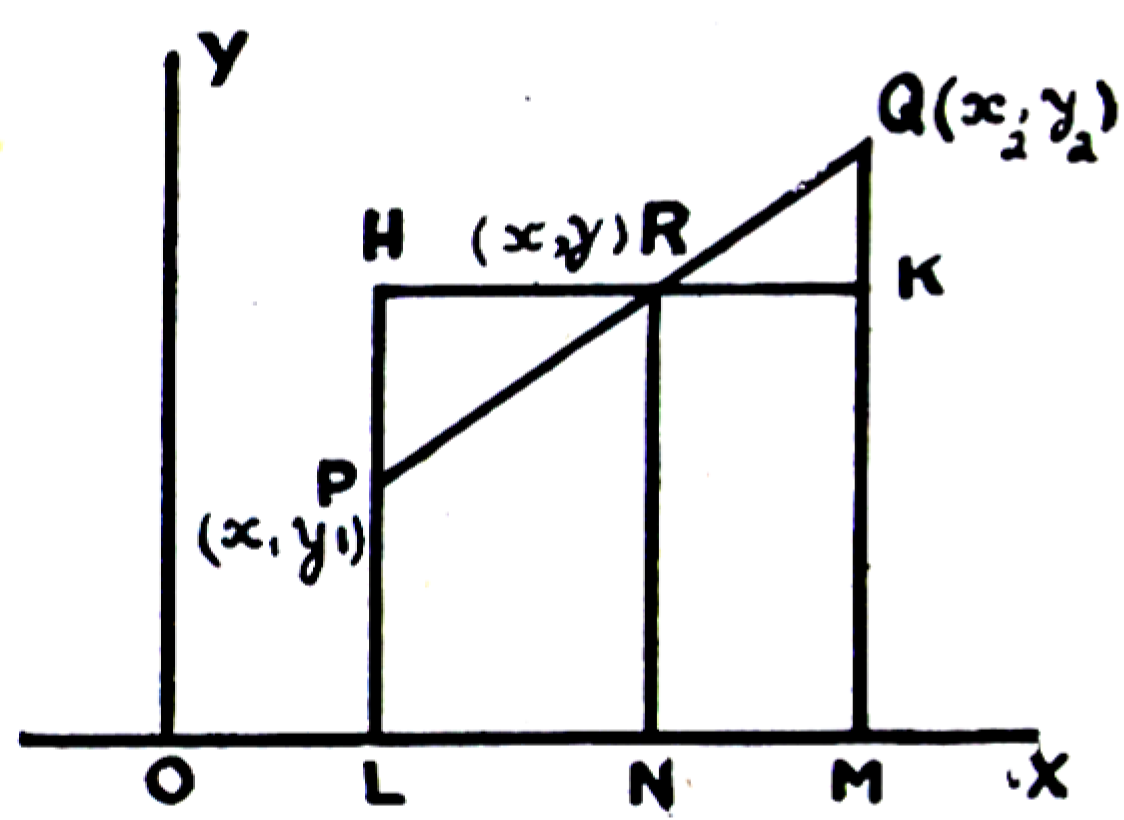
$$x^2+y^2-4x-6y-12=0.$$

10. If the distance of the point $(x, y)$ from $(a, o)$ be $(a+x)$, prove that $y^2=4ax$.

**1.4.** *To find the co-ordinates of a point which divides the line joining two given points in a given ratio internally.*

Let P $(x_1 y_1)$ and Q $(x_2 y_2)$ be the given points and let R $(x, y)$ be the point which divides PQ *internally* in the ratio $m : n$.

Draw PL, QM and RN perpendiculars to OX and HRK ∥ to OX meeting LP produced in H and MQ in K as in the figure.




The triangles PRH and QRK are similar.

$$\therefore \frac{HR}{RK}=\frac{PH}{KQ}=\frac{PR}{RQ}=\frac{m}{n}.$$

Now $HR=ON-OL=x-x_1$ $\quad RK=OM-ON=x_2-x$

$PH=NR-LP=y-y_1$ $\quad KQ=MQ-NR=y_2-y.$

$$\therefore \frac{x-x_1}{x_2-x}=\frac{y-y_1}{y_2-y}=\frac{m}{n}.$$

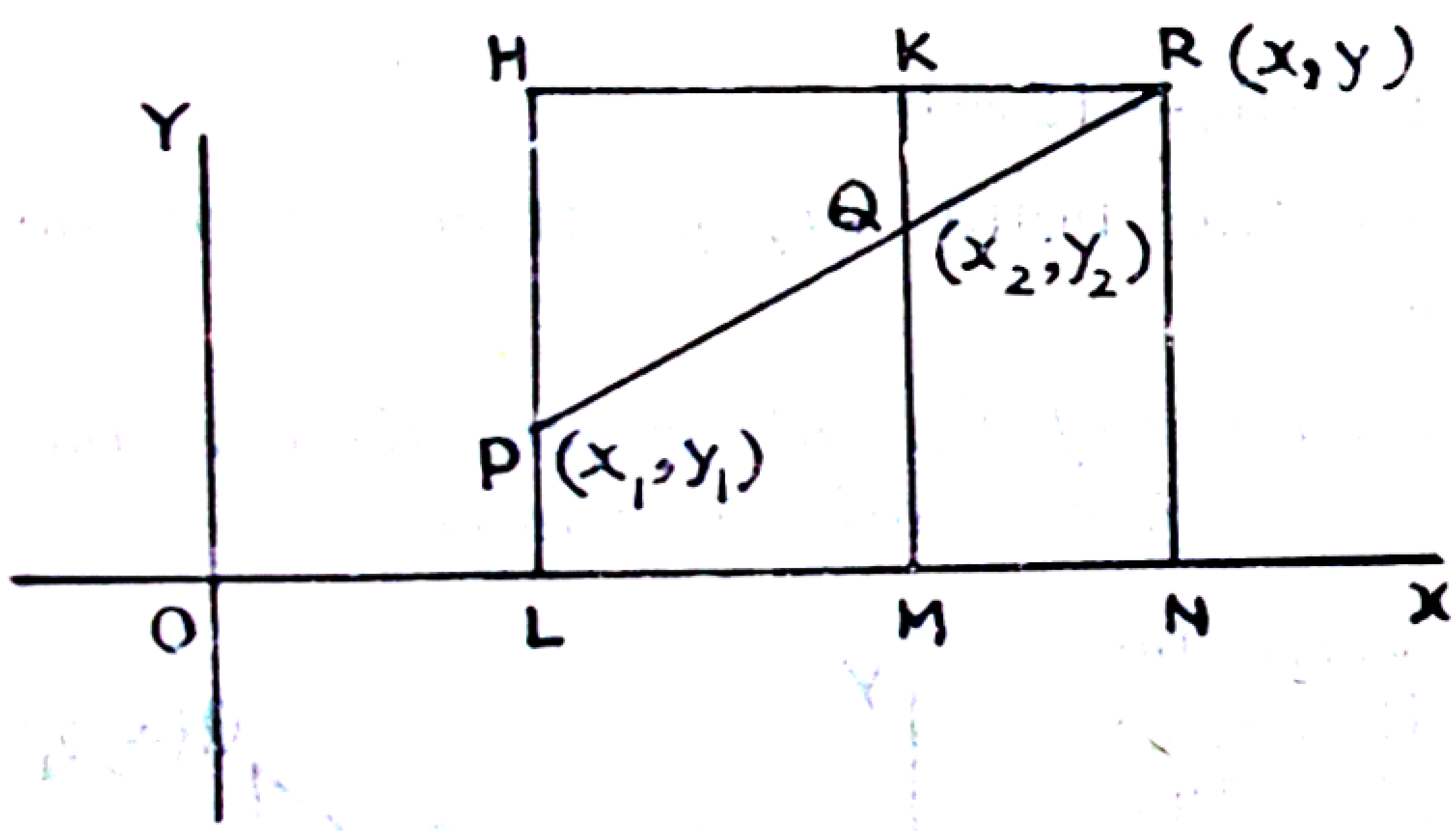
$$\therefore nx-nx_1=mx_2-mx$$

$$\therefore x(m+n)=mx_2+nx_1$$

$$\text{or } \mathbf{x}=\frac{\mathbf{mx_2+nx_1}}{\mathbf{m+n}}\ ;\ \mathbf{y}=\frac{\mathbf{my_2+ny_1}}{\mathbf{m+n}}.$$

**Cor.** The co-ordinates of the middle point of P,Q are given by $x=\dfrac{x_1+x_2}{2}$ ; $y=\dfrac{y_1+y_2}{2}$.

**1.41.** *To find the co-ordinates of the point which divides the line joining two given points externally in a given ratio.*




Let P $(x_1, y_1)$ and Q$(x_2, y_2)$ be the two given points.

Let R $(x, y)$ be the required point which divides PQ *externally* in the ratio $m : n$.

Draw PL, QM and RN perpendiculars to OX. Through R draw RKH parallel to OX meeting MQ and LP (both produced) in K and H respectively.

From similar triangles PRH and QRK, we have

$$\frac{HR}{KR}=\frac{PR}{QR}=\frac{m}{n}$$

But $HR=LN=ON-OL=x-x_1$

$KR=MN=ON-OM=x-x_2$

$$\therefore \qquad \frac{x-x_1}{x-x_2}=\frac{m}{n}$$

*i.e.* $nx-nx_1=mx-mx_2$

or $x(m-x)=mx_2-nx_1$

$$\therefore \qquad x=\frac{mx_2-nx_1}{m-n}$$

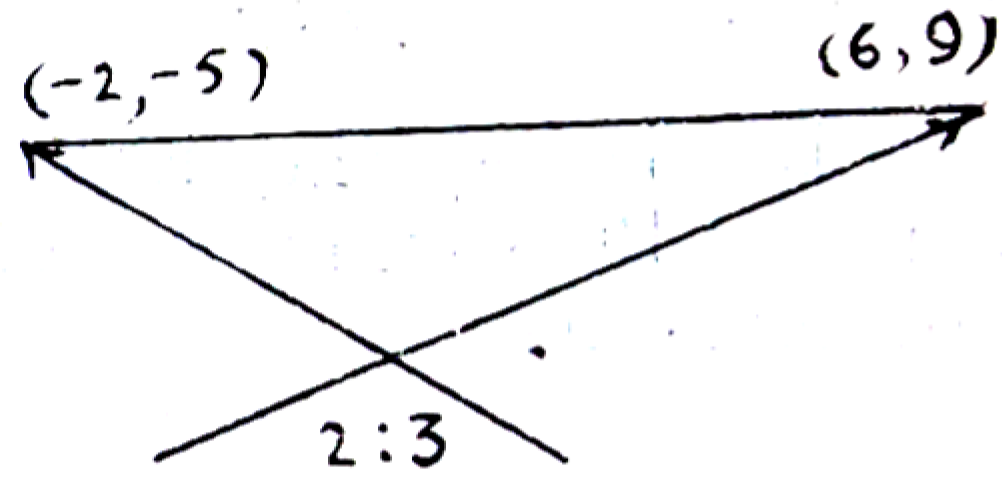
Similarly $y=\dfrac{my_2-ny_1}{m-n}$.

*Note.*—The formulae for external division of the join of two points may be deduced from the formulæ for internal division by changing $n$ into $-n$, for in this case, PR and RQ lie in the opposite directions and $\therefore \dfrac{PR}{RQ}=\dfrac{m}{-n}$

**Example 1.** *Find the co-ordinates af a point which divides the line joining the points $(-2, -5)$ and $(6, 9)$, (i) internally in the ratio $2 : 3$; (ii) externally in the ratio $3 : 1$.*

Let $(x, y)$ be the co-ordinates of the point of division.

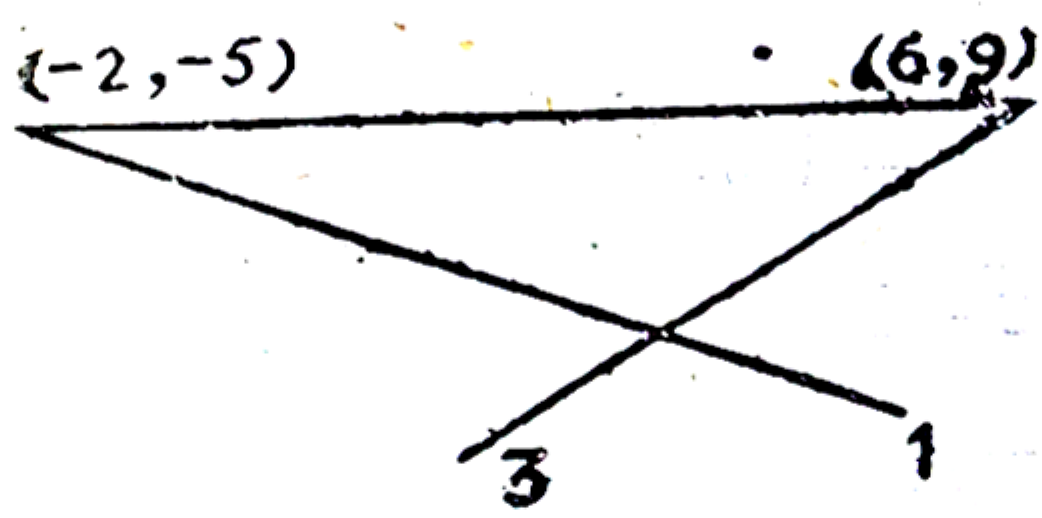
Then




(i) $x=\dfrac{2\times6+3(-2)}{2+3}=\dfrac{6}{5}$

$y=\dfrac{2\times9+3(-5)}{2+3}=\dfrac{3}{5}$

$\therefore$ the required point is $\left(\dfrac{6}{5}, \dfrac{3}{5}\right)$.




(ii) $x=\dfrac{3\times6-1(-2)}{3-1}=\dfrac{20}{2}=10$

$y=\dfrac{27-1(-5)}{3-1}=\dfrac{32}{2}=16$

$\therefore$ the required point is $(10, 16)$.

**Example 2.** *Find the co-ordinates of the middle point of the line joining the points $(-2, -5)$ and $(6, 9)$.*

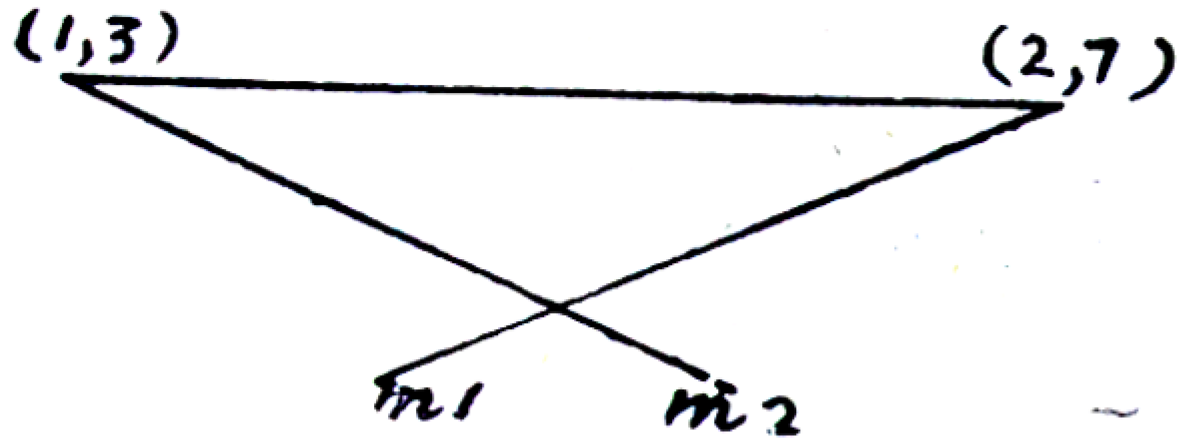
If $(x, y)$ be the middle point,

$x=\dfrac{-2+6}{2}=2$ ; $y=\dfrac{-5+9}{2}=2$.

$\therefore$ the middle point is $(2, 2)$.

**Example 3.** *In what ratio does the point $(\frac{10}{7}, \frac{33}{7})$ divide the line joining the point $(1, 3)$ and $(2, 7)$?*

Let the required ratio be




$m_1 : m_2$

$$\therefore \frac{10}{7} = \frac{2m_1+m_2}{m_1+m_2},$$

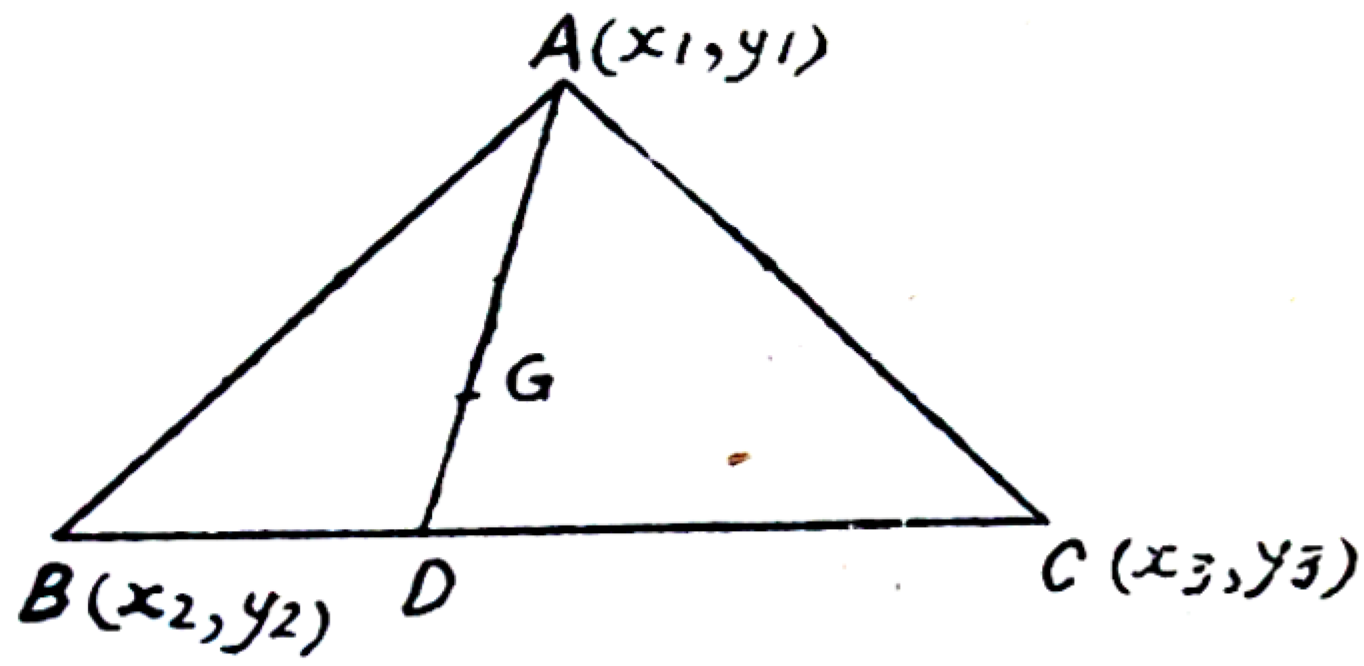
or $10m_1+10m_2=14m_1+7m_2.$

or $4m_1=3m_2 \quad \therefore \frac{m_1}{m_2}=\frac{3}{4}$

Hence the required ratio is 3 : 4.

**Example 4.** *To show that the medians of a triangle are concurrent.*

Let the vertices of the triangles ABC be A $(x_1, y_1)$, B$(x_2, y_2)$, C$(x_3, y_3)$.




Let D be the middle point of BC so that its co-ordinates are,

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

If G$(x, y)$ is any point on AD such that AG : GD=2 : 1, we have

$$x=\frac{2\frac{(x_2+x_3)}{2}+1(x_1)}{2+1}, \quad y=\frac{2\frac{(y_2+y_3)}{2}+1(y_1)}{2+1}$$

$$i.e., \; x=\frac{x_1+x_2+r_3}{3}, \quad y=\frac{y_1+y_2+y_3}{3},$$

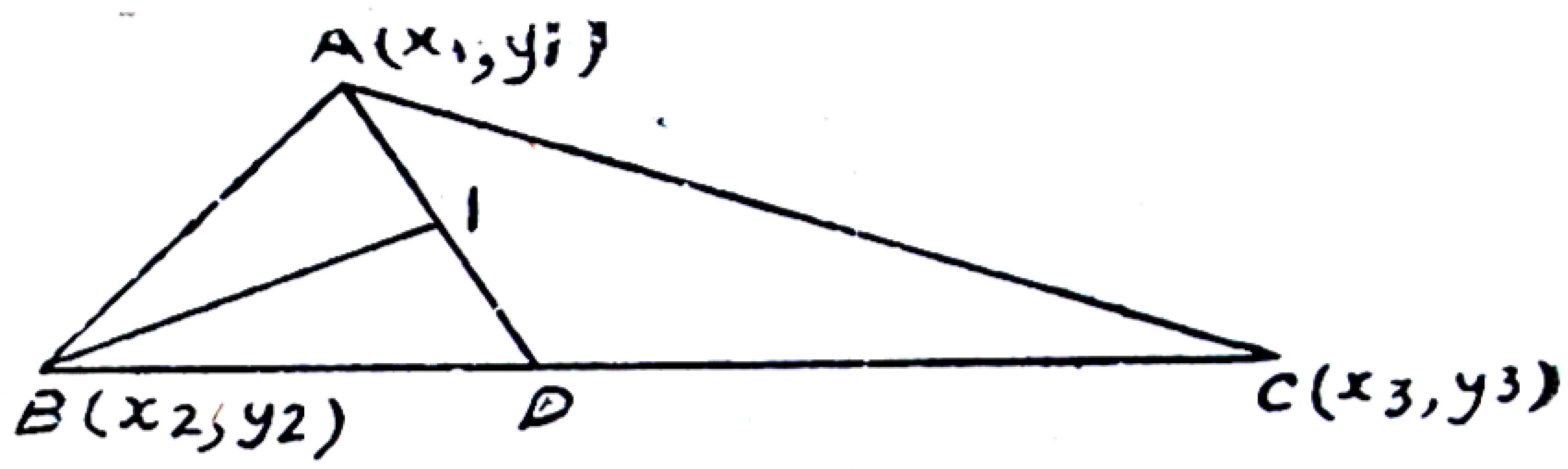
We observe that the co-ordinates of the point G, which divides the median AD in the ratio 2 : 1, are symmetrical in the co-ordinates of the vertices of the triangle (*i.e.*, remain unchanged if $x_1$ is changed into $x_2$, $x_2$ to $x_3$ and $x_3$ to $x_1$ and similarly about the three $y$'s). This shows that the co-ordinates of the points which divide the other two medians in the ratio 2 : 1 are the same as those of G, *i.e.* G lies on each median.

Hence the three medians are concurrent.

*Note.*—The point G is called the centroid of the triangle and its co-ordinates are $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$.

**Example 5.** *To show that the internal bisectors of the angles of a triangle are concurrent.* (*P.U.* 1944)

Let the vertices of the triangle ABC be A $(x_1, y_1)$, B$(x_2, y_2)$, C$(x_3, y_3)$ and let the lengths of the sides BC, CA, AB be $a$, $b$, $c$. Let AD, the bisector of the angle A, meet BC in D.




From geometry we know that the bisector AD divides the base BC in the ratio of the sides AB and AC.

$$\therefore \frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$$

*i.e.,* D divides BC internally in the ratio $c : b$.

$\therefore$ the point D is

$$\left(\frac{cx_3+bx_2}{c+b}, \frac{cy_3+by_2}{c+b}\right)$$

Let the bisector of the angle B meet AD in the point I.

$$\text{Now} \quad \frac{DC}{BD}=\frac{b}{c}, \quad \text{or} \quad \frac{DC}{BD}+1=\frac{b}{c}+1$$

$$\textit{i.e.,} \quad \frac{DC+CD}{BD}=\frac{b+c}{c} \quad \text{or} \quad \frac{a}{BD}=\frac{b+c}{c}$$

$$\therefore \qquad BD=\frac{ac}{b+c}.$$

As the bisector BI of the angle B divides the base AD in the ratio of the sides AB and BD, we have

$$\frac{AI}{ID}=\frac{BA}{BD}=\frac{c}{\dfrac{ac}{b+c}}=\frac{b+c}{a}$$

*i.e.,* I divides AD internally in the ratio $b+c : a$.

$\therefore$ The point I is

$$\left(\frac{(b+c)\left(\frac{cx_3+bx_2}{c+b}\right)+a(x_1)}{(b+c)+a}, \frac{(b+c)\left(\frac{cy_3+by_2}{c+b}\right)+a(y_1)}{(b+c)+a}\right)$$

*i.e.*, $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

The symmetry in the co-ordinates of the point I which is the point of intersection of the bisectors of the angles A and B, shows that the bisectors of the angles B and C meet in I, *i.e.*, I lies on each bisector.

Hence the internal bisectors of the angles of a triangle are concurrent.

*Note*:— The point I is called the **in-centre** of the triangle and its co-ordinates are $\left(\frac{\mathbf{ax_1+bx_2+cx_3}}{\mathbf{a+b+c}}, \frac{\mathbf{ay_1+by_2+cy_3}}{\mathbf{a+b+c}}\right)$

## Exercise I (c)

1. Find the co-ordinates of the middle points of the line joining the points :

(*i*) (2, 3) and (4, 5). (*ii*) (—4, —6) and (—2, 6). (*iii*) $(a+b, c+d)$ and $(a-b, c-d)$. (*P.U.* 1937)

2. (*i*) The vertices of a triangle are (2, 3), (4, —5), (—3, —6). Find the lengths of the medians.

(*ii*) A line is bisected at the origin and one end of the line is $(x_1, y_1)$. Find the co-ordinates of the other end.

(*iii*) The points (—2, 0), (4, 1), (1, 4) are the mid-points of the sides of a triangle. What are the co-ordinates of the vertices.

3. Find the co-ordinates of the point which divides the line joining the points :

(*i*) (3, 4) and (1, 2) internally in the ratio of 2 : 3.

(*ii*) (5, —2) and (3, —6) internally in the ratio of 1 : 3.

(*iii*) (—4, —1) and (2, 7) internally in the ratio of 3 : 5.

(*iv*) (—5, 0) and (—1, —3) externally in the ratio of 4 : 3.

(*v*) (4, 5) and (—3, —4) externally in the ratio of 7 : 1.

4. Find the ratio in which the line joining the points :

(*i*) (5, 7) and (−1, 3) is divided by (*a*) the $x$-axis (*b*) the $y$ axis.

(*ii*) (−3, +5) and (4, −9) is divided by the point (−2, 3).

5. The line joining (2, 3) and (4, −5) is trisected ; find the co-ordinates of the point of trisection nearest the former point.

6. Find the co-ordinates of the points that divide the line joining the points (−35, −20) and (5, −10) into four equal parts.

7. Find the co-ordinates of the centroid of the triangle whose vertices are :

(*i*) (3, −5); (−7, 4); (10, −2). (*P. U. 1939 Sup.*)

(*ii*) (4, 2) ; (4, 5) ; (−2, 2).

(*iii*) (8, 3) ; (−2, 3) ; (4, −5),

8. Find the co-ordinates of the in-centre of the triangle whose vertices are :

(*i*) (0, 0); (36, 15); (−20, 15).

(*ii*) (1, −2), (−2, 4); (3, −6).

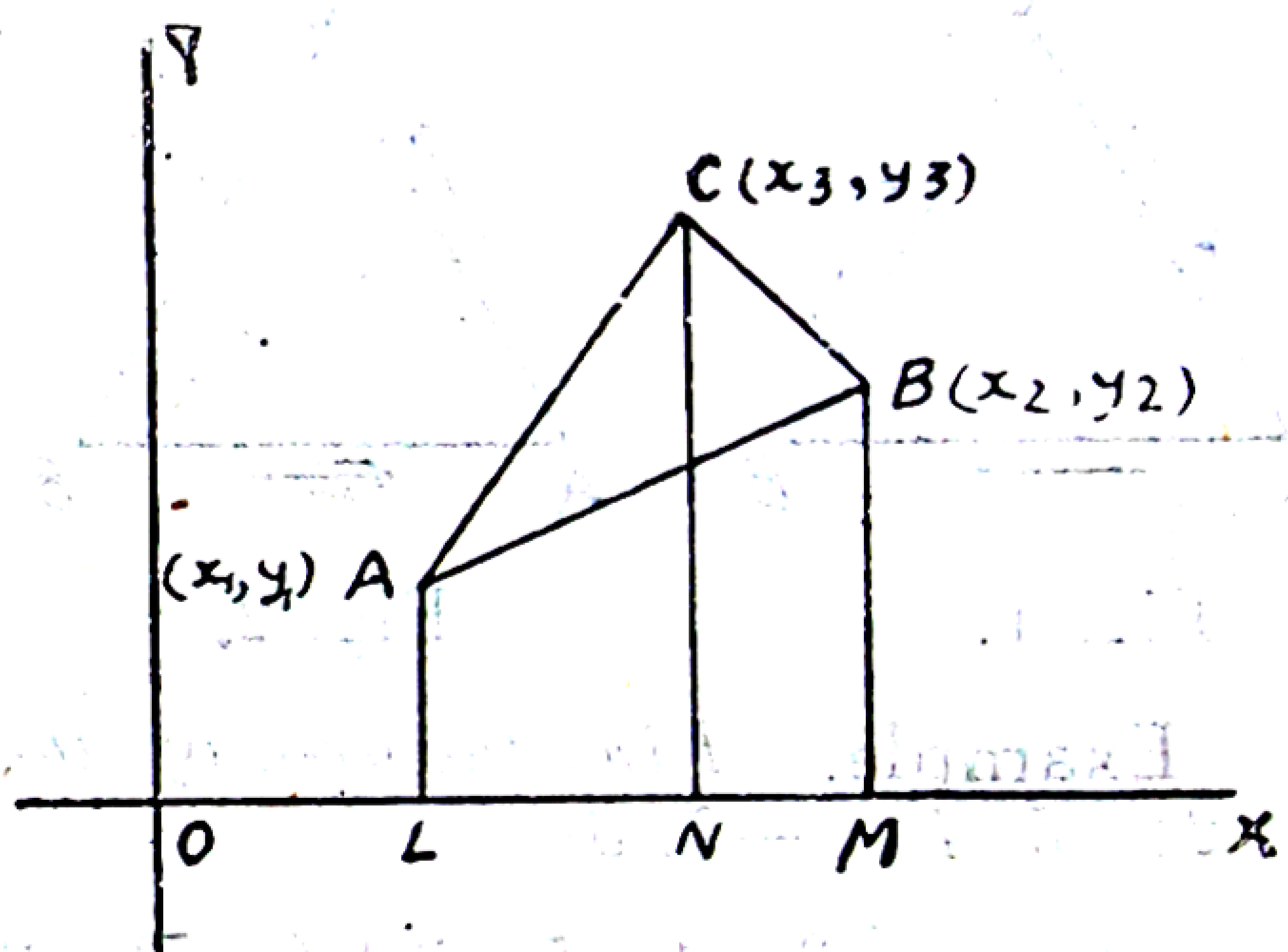
**1.5.** *To find the area of a triangle whose vertices are given.*

Let A $(x_1, y_1)$, B $(x_2, y_2)$, C $(x_3, y_3)$ be the vertices of the triangle.

Draw AL, BM, CN perpendiculars to OX

△ ABC = trapezium ALNC + trap. CNMB − trap. ALMB.

But the area of a trapezium = ½ (sum of the parallel sides) × perpendicular distance between them.




∴ trap. ALNC $= \frac{1}{2}(LA+NC)LN = \frac{1}{2}(y_1+y_3)(x_3-x_1)$,

trap. CNMB $= \frac{1}{2}(NC+MB)NM = \frac{1}{2}(y_3+y_2)(x_2-x_3)$,

$$\text{trap. ALMB}-\tfrac{1}{2}(\text{LA}+\text{MB})\text{LM}=\tfrac{1}{2}(y_1+y_2)\,(x_2-x_1),$$

$$\triangle\,\text{ABC}=\tfrac{1}{2}\left\{\,(y_1+y_3)\,(x_3-x_1)+(y_3+y_2)\,(x_2-x_3)\right.$$
$$\left.-(y_1-y_2)\,(x_2-x_1)\,\right\};$$

or omitting the terms which cancel,

$$\triangle\text{ABC}=\tfrac{1}{2}\left\{\,(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)\right.$$
$$=\tfrac{1}{2}\left\{\,(x_1y_2+x_2y_3+x_3y_1)-(x_2y_1+x_3y_2+x_1y_3)\,\right\}$$

**Cor.** The area of the triangle formed by the points $(x_1, y_2)$; $(x_2, y_2)$ and $(0, 0)$ is $\frac{1}{2}\,(x_1y_2-y_1x_2)$.

*Notes* : **1**. The following is the working rule for putting down the area of a triangle whose vertices are given :

(1) *Write down the co-ordinates of the three vertices in order in two columns, abscissae in one, ordinates in the other, repeating the co-ordinates of the first vertex.* (2 *Multiply the abscissa of each row with the ordinate of the next and add the products. This gives* $x_1y_2+x_2y_3+x_3y_1$. (3) *Multiply each ordinate by the abscissa of the next row and add the products. This givest* $x_2y_3+x_3y_1+x_1y_2$. (4) *Subtract the result in* (3) *from the result in* (2) *and divide by 2. This gives us the required area.*

2. **Sign of the area of a triangle.**

The above expression for the area of a triangle gives a positive result if in passing round the perimeter of triangle in the order of the vertices A, B, C, the area is always on the left hand or if the order of the description of the circuit ABCA is counterclock-wise (Fig. 1).

If the area is on the right hand or if the order is clockwise (Fig. 2), the above expression gives a negative result.

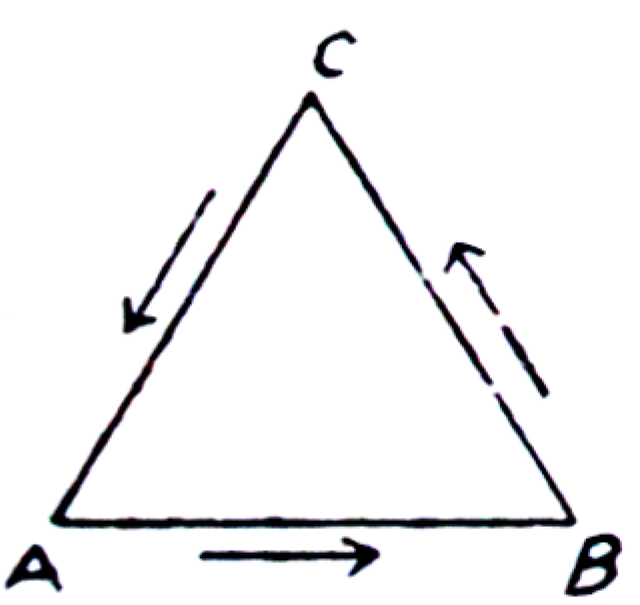



Fig. 1.

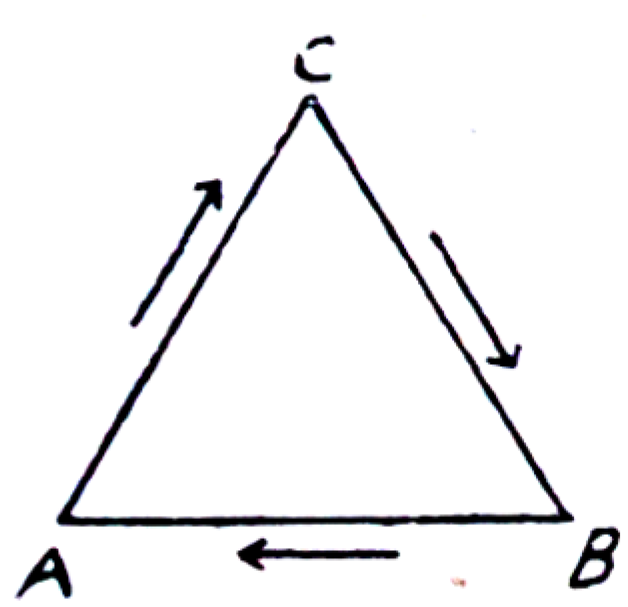



Fig. 2.

**Example.** *Find the area of the triangle whose vertices are* $(3, 2)$; $(5, 4)$; $(-7, 3)$

$$\triangle=\tfrac{1}{2}\left\{\,3\times4+5\times3+(-7)\times2-2\times5-4\times(-7)-3\times3\,\right\}$$
$$=\tfrac{1}{2}\left\{\,12+15-14-10+28-9\,\right\}$$

### 1.51. Condition of collinearity of the points.

The points A $(x_1, y_1)$, B $(x_2, y_2)$, C $(x_3, y_3)$ will be in one straight line if the area of the triangle ABC is zero. The condition for this is

$$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0.$$

**Example.** *Show that the points $(a, 0)$ ; $(0, b)$; $(3a, -2b)$ lie in a straight line.* (*P.U. 1936*)

| | |
|---|---|
| $a$ | $0$ |
| $0$ | $b$ |
| $3a$ | $-2b$ |
| $a$ | $0$ |

Area of the triangle formed by the three points

$= \frac{1}{2} \{ ab + 0 + 0 - 0 - 3ab + 2ab \}$

$= 0.$

$\therefore$ the three points lie on a straight line.

## Exercise I (d)

1. Find the area of the triangle whose vertices are :

(*i*) $(0, 0)$, $(2, 6)$, $(4, 4)$ ; (*ii*) $(2, -5)$, $(2, 8)$, $(-2, 0)$ ; (*iii*) $(8, 3)$, $(-2, 3)$, $(4, -5)$ ; (*iv*) $(3, 6)$, $(2, 10)$, $(1, 3)$ ; (*v*) $(4, 1)$ $(0, 3)$, $(2, 1)$ ; (*vi*) $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$.

2. Show that the following points are collinear :

(*i*) $(-4, -5)$, $(1, -1)$, $(6, 3)$ ; (*ii*) $(1, 4)$, $(3, -2)$, $(-3, 16)$; (*iii*) $(2, 3)$ $(-1, -2)$, $(5, 8)$. (*P.U.*)

3. If $(x, y)$ is any point on the line joining the points $(7, 0)$ and $(0, 5)$ show that

$$\frac{x}{7} + \frac{y}{5} = 1.$$

4. If $(x, y)$ is any point on the line joining the points $(2, 5)$ and $(5, 7)$ show that $2x - 3y + 11 = 0$.

5. If $(x, y)$ is any point on the line joining the points $(3, 4)$ and $(-1, 2)$ show that $x - 2y + 5 = 0$.

6. Prove that the middle point of the hypotenuse of a right-angled triangle is equidistant from the vertices. (*P.U. 1948*)

## CHAPTER II

# LOCUS AND ITS EQUATION

**2.1. Def.** *When a point moves in a plane under some given geometrical condition, its path in the plane is called its* **locus.**

Thus, for example

1. If a point moves so as to keep at a fixed distance from a given point its locus is a *circle.*

2. If a point moves so that its distances from two fixed points are always equal to one another, its locus is the right bisector of the straight line joining the two points.

**2.2.** When a point traces out a locus under some given geometrical condition, there will be some algebraical relation which is satisfied by the co-ordinates of all the points on the locus, and by the co-ordinates of no other point. This algebraical relation is called the **equation of the locus.**

Conversely all points whose co-ordinates satisfy a given algebraical equation lie on a curve, which is called the locus of that equation.

*It should be noted that a point lies upon a curve when and only when its co-ordinates satisfy the equation of the locus.*

The co-ordinates of the moving point are generally denoted by $(x, y)$ and are called the **current** co-ordinates. Thus the equation of the locus will involve $x$, $y$ and given quantities.

**2.21.** The following process has generally to be followed for finding the equation of the locus of a point moving under a given geometrical condition :

*First step. Assume that $P(x, y)$ is any point on the locus.*

*Second step. Write down the given geometrical condition.*

*Third step. Express the condition algebraically and simplify the result. The final equation, containing $x$, $y$, and the given quantities will be the required equation.*

**Example 1.** *A point moves so that its distance from the $x$-axis is 4 times its distance from the $y$-axis. Find the equation of the locus.*

1. Let P $(x, y)$ be any point on the locus.

2. If PM be drawn perpendicular from P to OX, the condition under which the point P moves is $MP=4.OM$.

3. But $MP=y$ and $OM=x$

$\therefore$ the required equation of the locus is

$$y=4x.$$

**Example 2.** *A point moves so that it is equidistant from points $A\ (-4, 0)$ and $B\ (2, 1)$. Find its locus.*

1. Let P $(x, y)$, be any point on the locus.

2. The condition under which the point P moves is $PA=PB$.

3. Now $PA=\sqrt{(x+4)^2+y^2}$ and $PB=\sqrt{(x-2)^2+(y-1)^2}$. Equating these, we have

$$\sqrt{(x+4)^2+y^2}=\sqrt{(x-2)^2+(y-1)^2},$$

or $(x+4)^2+y^2=(x-2)^2+(y-1)^2$,

*i.e.* $12x+2y+11=0$,

which is the required equation of the locus.

**Example 3.** *A point moves so that its distance from the point $A\ (4, 3)$ is always equal to 5. Find the equation of the locus and show that it passes through the origin.*

1. Let P $(x, y)$ be any point on the locus.

2. The condition under which the point P moves is $PA=5$.

3. But $PA=\sqrt{(x-4)^2+(y-3)^2}$

$\therefore$ $\sqrt{(x-4)^2+(y-3)^2}=5$,

or $(x-4)^2+(y-3)^2=25$,

*i.e.*, $x^2+y^2-8x-6y=0$ ...(1)

which is the required equation of the locus.

The co-ordinates of the origin are (0, 0). If we put $x=0$ and $y=0$ in the L.H.S. of (1) the result is zero, which shows that the equation of the locus is satisfied by the co-ordinates of the origin.

$\therefore$ the origin lies on the locus represented by (1).

**Example 4.** *A point moves so that the sum of the squares of its distances from two fixed points A (1, 0) and B (–1, 0) is constant and equal to $2a^2$; find the equation of the locus.*

1. Let P $(x, y)$ be any point on the locus.
2. The condition under which the point P moves is $PA^2+PB^2=2a^2$..............(1)
3. Now $PA^2=(x-1)^2+y^2$ and $PB^2=(x+1)^2+y^2$

   $\therefore$ (1) becomes $(x-1)^2+y^2+(x+1)^2+y^2=2a^2$,

   or $x^2+y^2=a^2-1$,

   which is the required equation of the locus.

## Exercise II

1. A point moves such that its distance from the point (3, 2) is 5. Find its locus.

2. Find the equation of the locus of a point which moves in such a way that

   (*i*) its distance from $x$-axis is equal to two times its distance from $y$-axis.

   (*ii*) three times its distance from $x$-axis is equal to four times its distance from $y$-axis.

3. A point moves so that its distance from $x$-axis is equal to its distance from the point (2, 3). Find the equation of its locus.

4. Find the locus of a point which moves so that its distance from the point (3, 0) is three times its distance from (0, 3).

5. Find the equation of the locus of a point which moves so that

   (*i*) sum of the squares of its distances from (0, 3) and (2, 1) is always equal to 6.

(*ii*) the difference of the squares of its distances from the points $(3, 0)$ and $(-3, 0)$ is 7.

(*iii*) The sum of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$. (*P. U. 1944*)

6. Find out which of the points $(0, 0)$, $(2, 1)$, $(2, 0)$, $(\sqrt{5}, 0)$, $(-2, -1)$ lie on the locus whose equation is $x^2+y^2=5$.

7. Show that the locus of the point whose distance from $(-ae, 0)$ is $(ex+a)$ has the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)}=1$.

8. Show that for all values of $\theta$, the point $(a \cos \theta, b \sin \theta)$ lies on the locus $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

9. When will the locus $ax+by+c=0$ pass through the origin.

10. Find the co-ordinates of the points where the locus represented by $x^2+y^2=9$ cuts the axes of co-ordinates.

## CHAPTER III

# THE STRAIGHT LINE

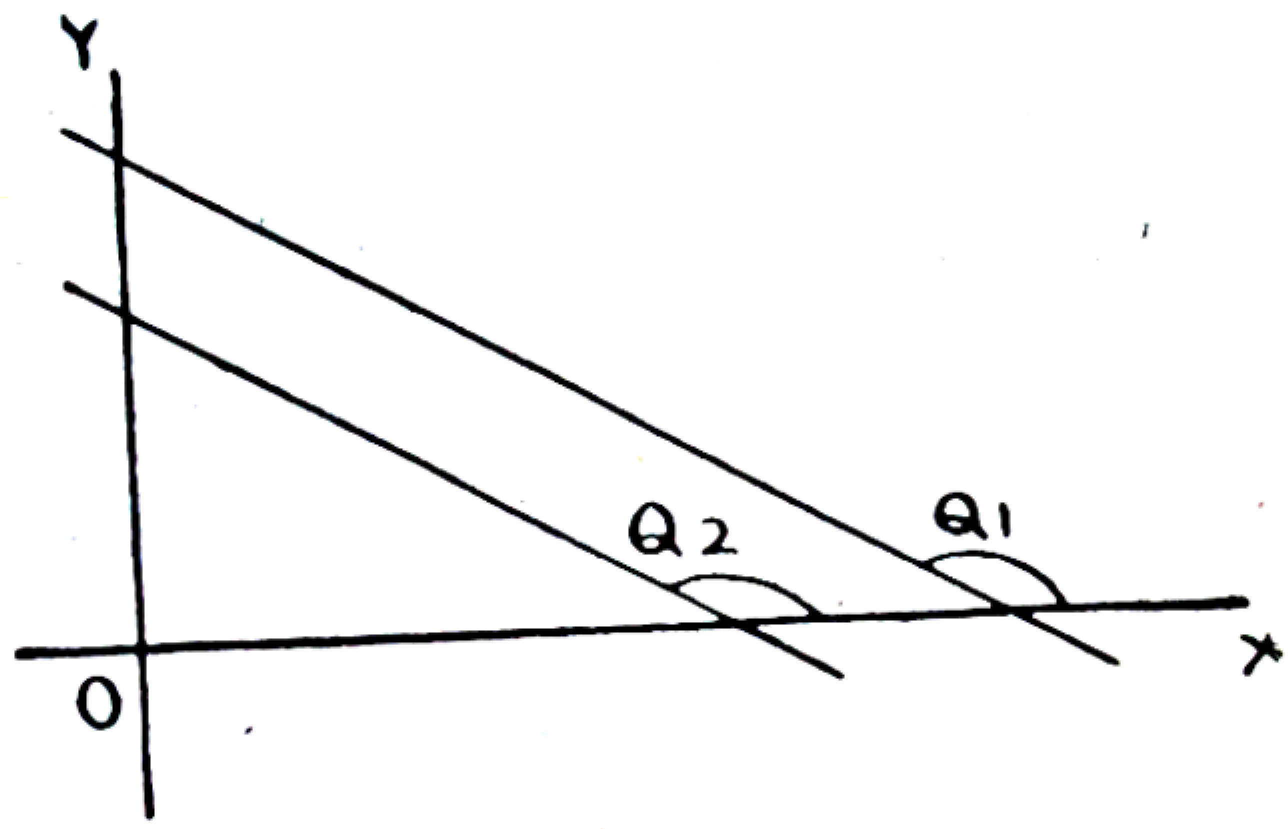
**3.1. Slope of a line.** *The tangent of the angle which a straight line makes with the positive direction of x-axis is called its* **slope** *or* **gradient**.

The slope of a line is generally denoted by $m$.

**3.11.** Let $m_1$, $m_2$ be the slopes of two lines which are given to be parallel. Because the lines are parallel, their inclinations $\theta_1$ and $\theta_2$ are equal.

$\therefore$ $\tan\theta_1 = \tan\theta_2$

or $m_1 = m_2$




*Conversely* if $m_1 = m_2$, then $\tan\theta_1 = \tan\theta_2$.

$\therefore$ $\theta_1 = \theta_2$ (both lying between 0° and 180°)

*i.e.*, the two lines are parallel.

$\therefore$ *The slopes of two parallel lines are equal and conversely.*

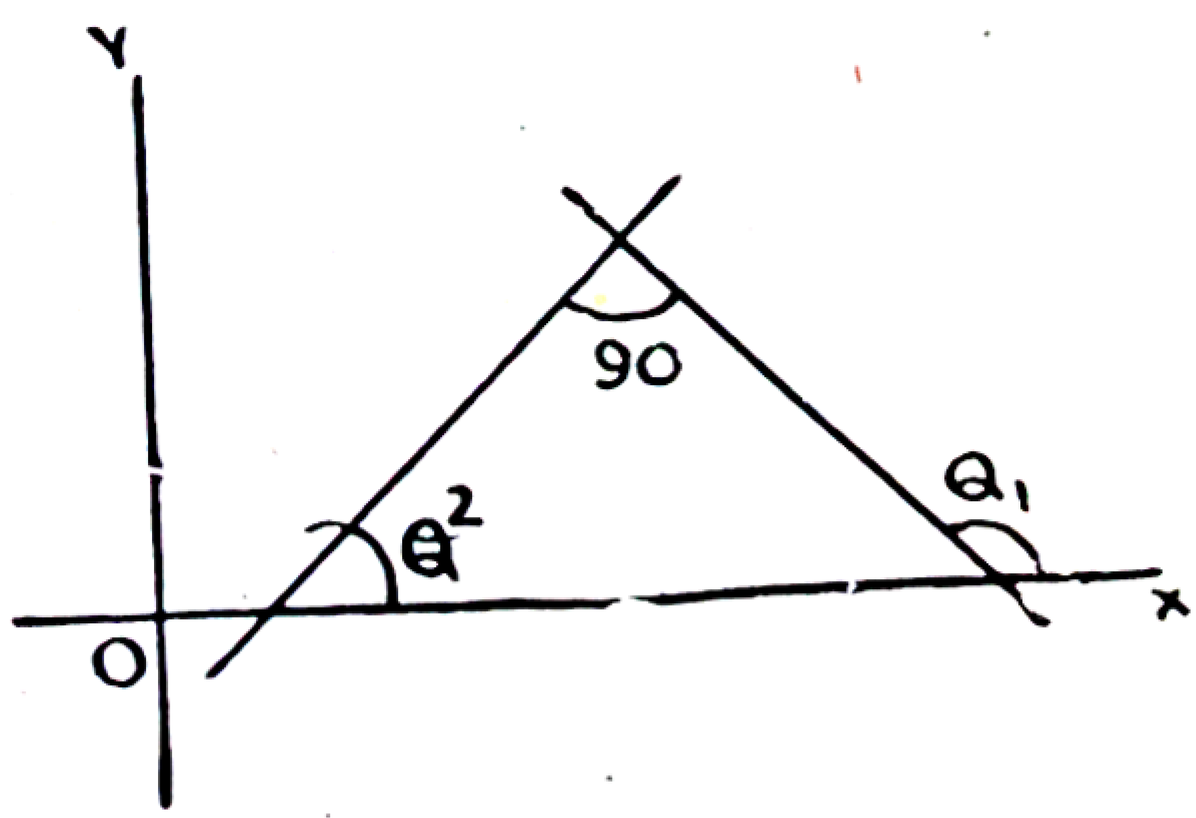
**3.12.** Let there be two lines perpendicular to each other. Let their inclinations be $\theta_1$, $\theta_2$ and $m_1$, $m_2$ their slopes.

Now $\theta_1 = 90° + \theta_2$

or $\tan\theta_1 = \tan(90° + \theta_2)$

$$= -\cot\theta_2$$

$$= -\frac{1}{\tan\theta_2}$$




or $m_1 = -\dfrac{1}{m_2}$ or $m_1 m_2 = -1$.

Conversely if $m_1 = -\frac{1}{m_2}$

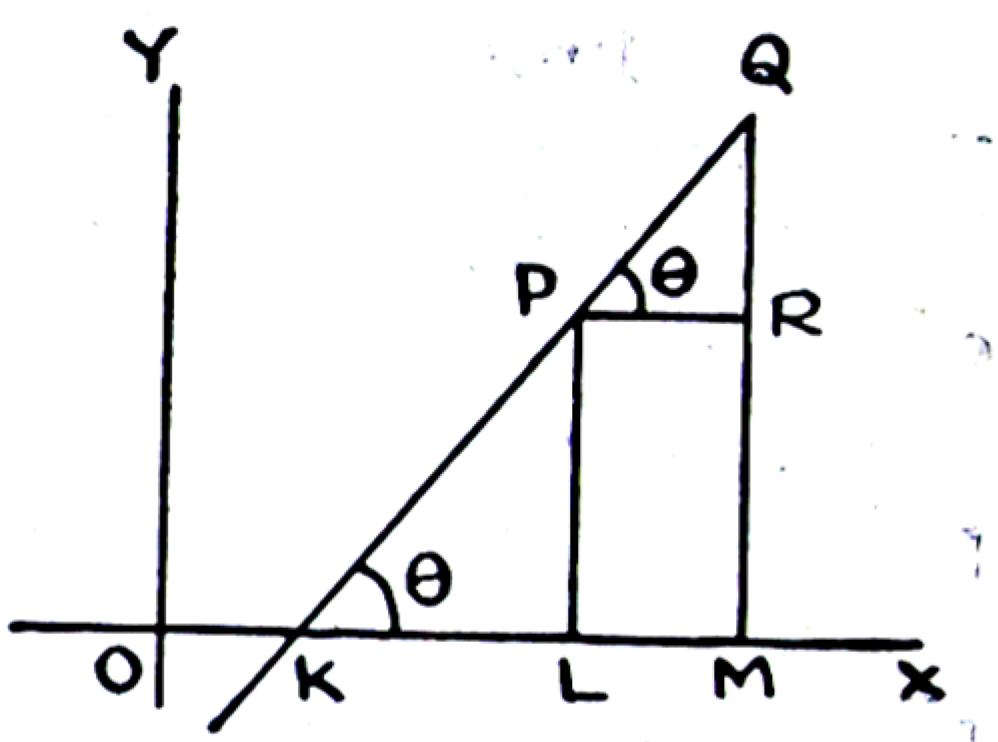
$$i.e., \tan \theta_1 = -\frac{1}{\tan \theta_2} = -\cot \theta_2$$
$$= \tan(90° + \theta_2)$$
$$\therefore \quad \theta_1 = 90° + \theta_2$$

which shows that the two lines are perpendicular to each other.

*Hence two lines are perpendicular to each other if the product of their slopes is equal to $-1$ or if the slope of each is the negative reciprocal of the other, and conversely.*

### 3.13. Slope of the line joining two points.

Let the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ make an angle $\theta$ with $x$-axis.




Draw PL and QM perpendiculars to OX. Draw PR perpendicular to QM.

Let QP cut OX in K.
$\angle QPR = \angle QKX = \theta$
Also $PR = LM = OM - OL$
$$= x_2 - x_1$$
and $RQ = MQ - MR = MQ - LP = y_2 - y_1$

Now in the right-angled triangle PRQ

$$\tan \theta = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$i.e., m = \frac{y_2 - y_1}{x_2 - x_1}$$

which gives the slope of the line in terms of the co-ordinates of the given points.

**Example.** *Show that the points $A(3, 2)$, $B(11, 8)$, $C(8, 12)$ and $D(0, 6)$ are the vertices of a rectangle.*

$$\text{Slope of AB} = \frac{8-2}{11-3} = \frac{6}{8} = \frac{3}{4}$$

$$\text{Slope of BC} = \frac{12-8}{8-11} = -\frac{4}{3}$$

$$\text{Slope of CD}=\frac{6-12}{0-8}=\frac{-6}{-8}=\frac{3}{4}$$

$$\text{Slope of DA}=\frac{6-2}{0-3}=\frac{-4}{3}$$

*i.e.*, AB $\parallel$ DC and BC $\parallel$ AD

Also AB $\perp$ BC

This shows that ABCD is a rectangle.

## Exercise III (a)

1. Show that the following sets of points are collinear :

(*i*) (2, 3), (−1, −2), (5, 8).

(*ii*) (1, 4), (3, −2), (−3, 16).

2. Prove that the points (2, −2), (5, 2), (−2, 1) are the vertices of a right-angled triangle.

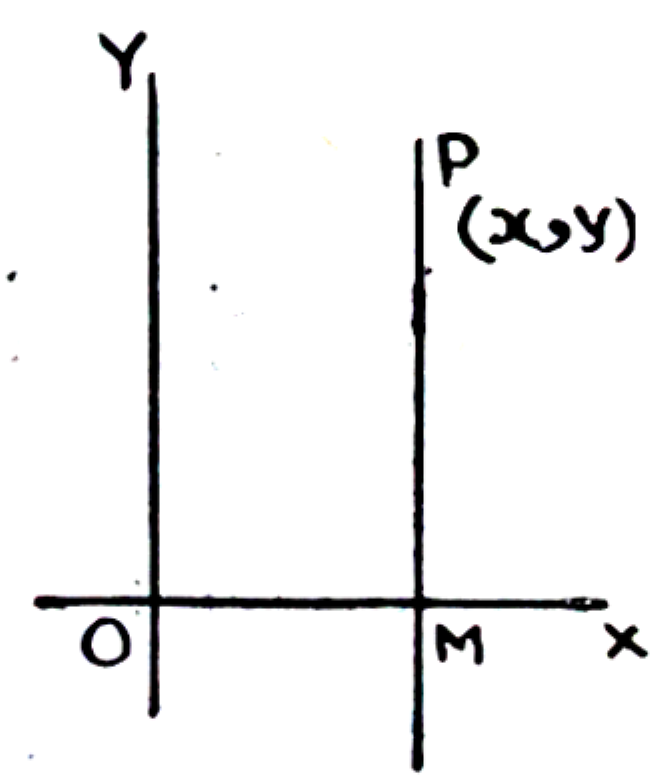
3. F nd the slopes of (*i*) the medians, (*ii*) the altitudes of the ab ve triangle.

4. Prove that in any triangle the line joining the mid-points of the sides is parallel to and half of the base.

5. Find the slopes of the sides and diagonals of the quadrilateral whose vertices are (1, 4), (2, −3), (8, −2), (−3, 6).

## Different Forms of the Equation of a Straight Line

**3.2. Straight lines parallel to the axes.** Let PM be a straight line parallel to $y$-axis and cutting the axis of $x$ in M. Such that OM$=a$.




Let P($x$, $y$) be any point on the line.

Then $x=$OM$=a$

$\therefore$ $x=a$ is the equation of the line.

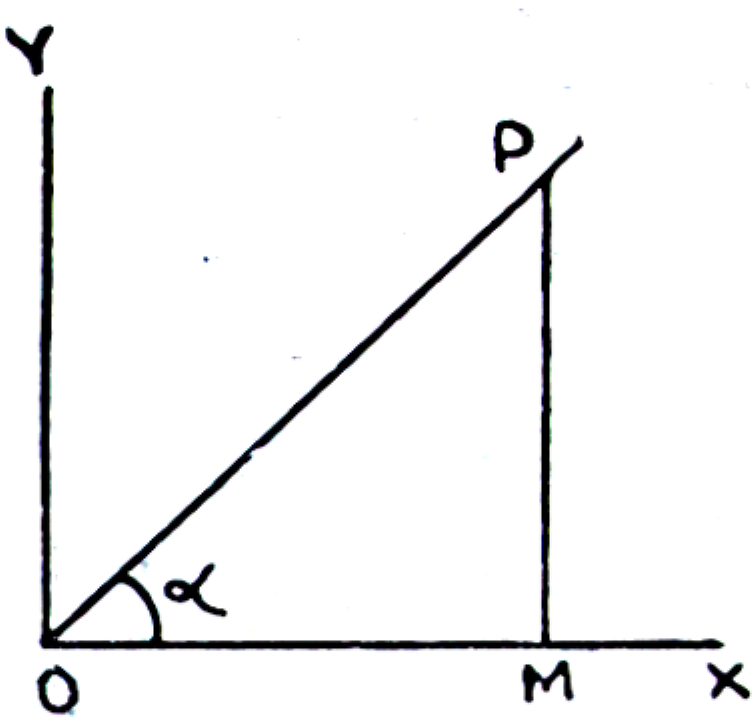
**Cor.** If $a=0$, the line MP coincides with the axis of $y$.

Hence $x=0$ is the equation of $y$-axis. Similarly it can be shown that a straight line parallel to $x$-axis is represented by $y=b$ and that the equation of the $x$-axis is $y=0$.

**3.21.** *To find the equation of a straight line which passes through the origin.*

Let QP be the straight line through O which makes an angle $\alpha$ with OX.

Take $P(x, y)$ any point on the line. Draw PM perpendicular to the $x$-axis.




From the triangle MOP,

$$\tan\alpha = \frac{MP}{OM} = \frac{y}{x}$$

$\therefore$ $y = x\tan\alpha$ is the equation of the line.

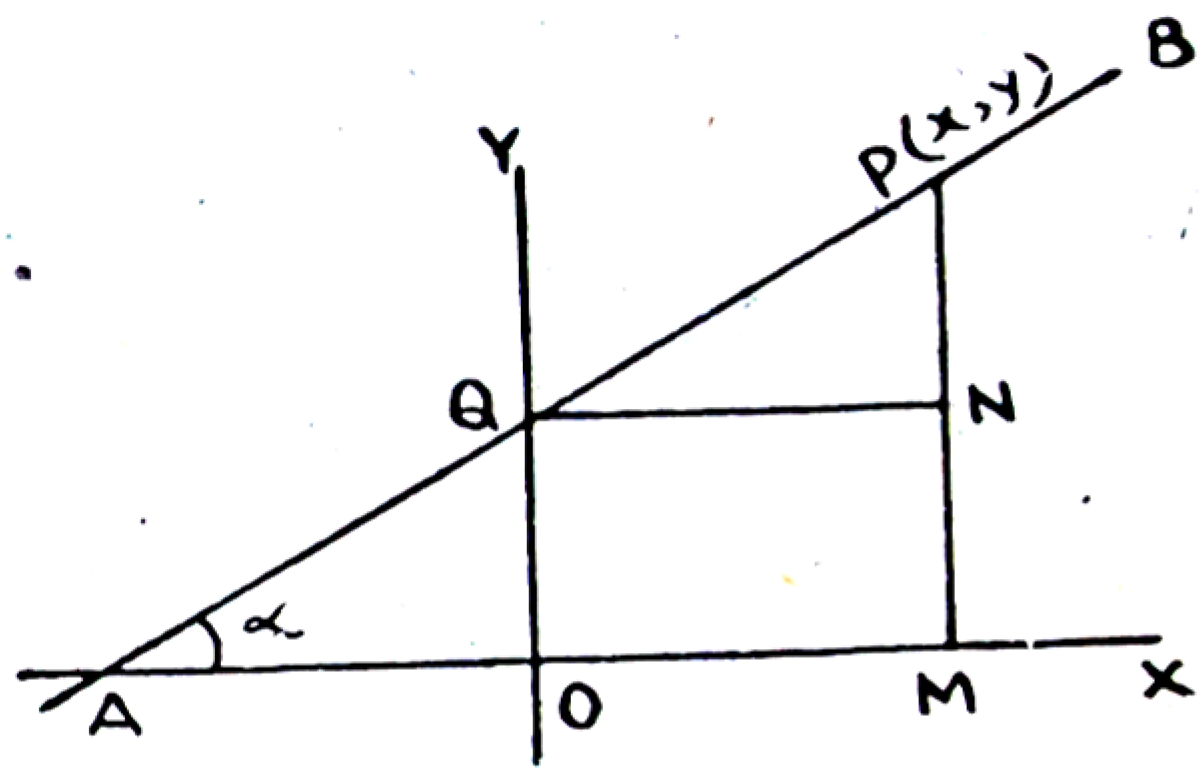
Writing $m$ for $\tan\alpha$ the equation of the lines becomes

$$y = mx.$$

**3.22.** *To find the equation to a straight line which cuts off a given intercept on the axis of $y$ and is inclined at a given angle to the axis of $x$.*

Let AB the given line, which cuts an intercept $OQ = c$ on $y$-axis and makes an angle $\alpha$ with OX.

Take $P(x, y)$ any point on the line. Draw PM $\perp$ to OX and QN parallel to OX to meet MP in N.




Now $\angle PQN = \angle QAX = \alpha$

From the $\triangle PQN$

$$\tan\alpha = \frac{NP}{QN} = \frac{MP - MN}{OM}$$

or $\quad \tan\alpha = \dfrac{y - c}{x}$

or $\quad y - c = x\tan\alpha$

$\therefore \quad y = x\tan\alpha + c$

is the equation of the straight line or

$$y = mx + c \qquad \ldots(i)$$

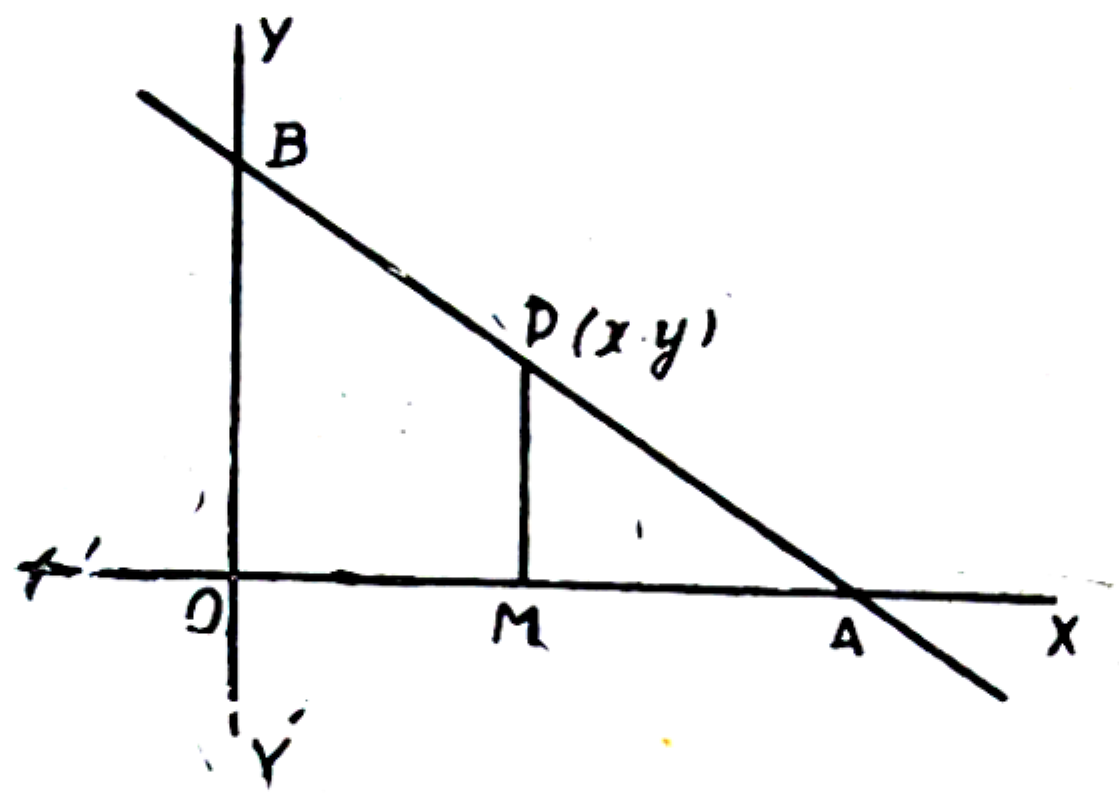
where $m$ is the slope of the line.

**Cor.** If $c = 0$, the line passes through the origin. Hence the equation of any line passing through the origin is $y = mx$.

**Notes.** 1. This form of the equation of a straight line is called the **tangent** or the **slope form**.

2. By giving suitable values to $m$ and $c$ the equation $y=mx+c$ may be made to represent any straight line.

**3.23.** *To find the equation of a straight line which cuts off given intercepts a and b from the axes.*




Let the straight line AB cut the axes at A and B such that OA$=a$ and OB$=b$.

Take any point P $(x, y)$ on the straight line AB.

Draw PM perpendicular to OX.

Now the triangles AMP and AOB are similar.

$$\therefore \quad \frac{MA}{OA}=\frac{MP}{OB}.$$

We have MP$=y$. OM$=x$.

$$\therefore \quad \frac{OA-OM}{OA}=\frac{MP}{OB}.$$

$$\therefore \quad \frac{a-x}{a}=\frac{y}{b},$$

$$\text{or} \quad \frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}.$$

This is the required equation, for it is the relation that holds between the co-ordinates of any point on the line.

This equation can be found in the following way also.

The sum of the areas of the triangles OPB and OPA is equal to the area of the triangle AOB.

$$\therefore \quad \tfrac{1}{2}ay+\tfrac{1}{2}bx=\tfrac{1}{2}ab$$

$$\therefore \quad \frac{x}{a}+\frac{y}{b}=1.$$

**Note.** This form of the equation of the straight line is called the **intercept form**.

**Example 1.** *Find the equation of a straight line which makes an angle of 60° with OX and cuts OY at a distance 7 units from O.*

Here $\quad m = \tan 60 = \sqrt{3}$ and $c = 7$

$\therefore$ the required equation is

$$y = \sqrt{3}\,x + 7.$$

**Example 2.** *Find the equation of a straight line which passes through the point (4, 5) and makes equal intercepts on the axis.*

Let the intercepts be each equal to $a$.

Then the equation of the straight line is

$$\frac{x}{a} + \frac{y}{a} = 1$$

$\because$ the straight line passes through the point (4, 5) these co-ordinates must satisfy the equation.

$$\therefore \qquad \frac{4}{a} + \frac{5}{a} = 1$$

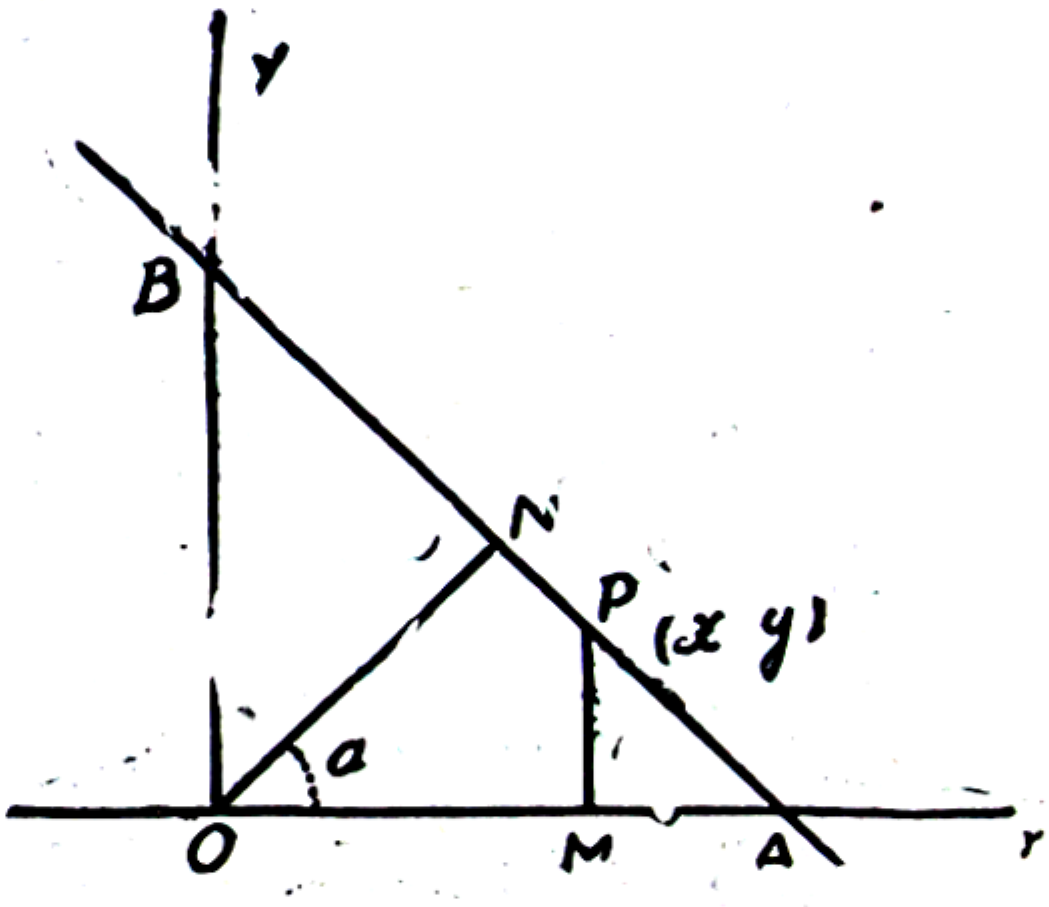
or $$a = 9$$

$\therefore$ required equation is

$$\frac{x}{9} + \frac{y}{9} = 1$$

or $$x + y = 9.$$

**3.24.** *To find the equation to a straight line in terms of the length of the perpendicular upon it from the origin and the angle which that perpendicular makes with the axis of x.*




Let ON be the perpendicular from O and let its length be $p$. Let $\alpha$ be the angle that ON makes with OX.

Let P be any point whose co-ordinates are $(x, y)$ lying on AB.

Draw PM perpendicular on OX.

Then

$$ON = OA \cos \alpha$$

$$=(\text{OM}+\text{MA})\cos\alpha$$
$$=\text{OM}\cos\alpha+\text{MA}\cos\alpha$$

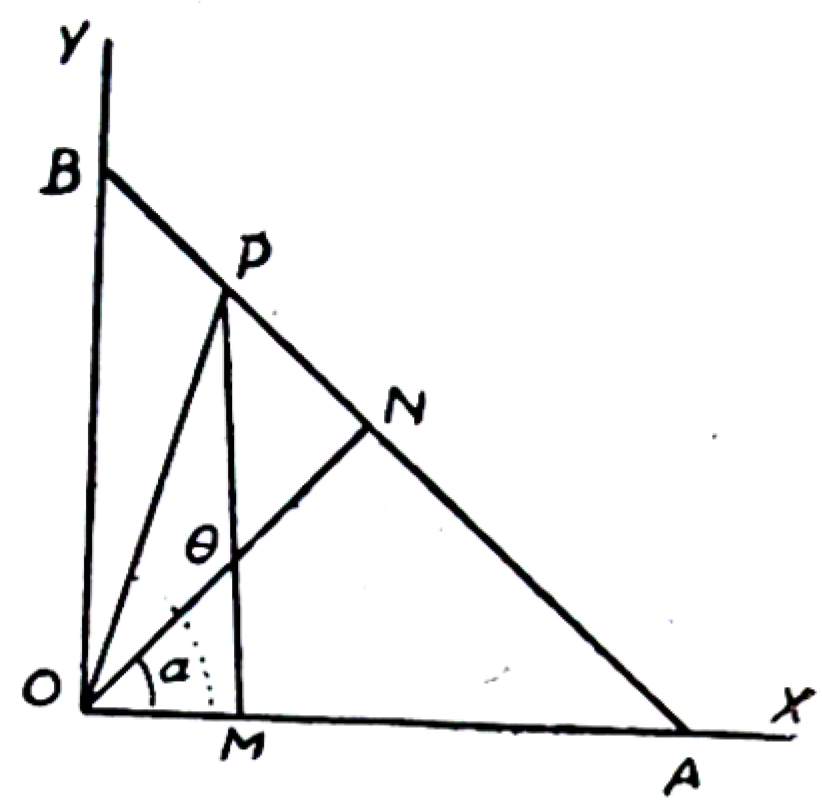
But $\dfrac{\text{MA}}{\text{MP}}=\tan\alpha$ $\quad\therefore\quad$ $\text{MA}=\text{MP}\tan\alpha.$

$$\therefore\quad \text{ON}=\text{OM}\cos\alpha+\text{MP}\tan\alpha\cos\alpha$$
$$=\text{OM}\cos\alpha+\text{MP}\sin\alpha$$

$\therefore$ we have $\mathbf{x\cos\alpha+y\sin\alpha=p}$.

This is the required equation.

Otherwise :—




Let $\angle$XOP$=\theta$, then

$x=\text{OM}=\text{OP}\cos\theta$

$y=\text{MP}=\text{OP}\sin\theta$

In the triangle ONP

$p=\text{ON}=\text{OP}\cos(\theta-\alpha)$

$\quad=\text{OP}(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$

$\therefore\quad p=x\cos\alpha+y\sin\alpha.$

**Note** 1. This form of the equation of the straight line is called the **perpendicular** or the **normal** form.

2. In this form sum of the squares of the coefficients of $x$ and $y$ is equal to 1.

## Exercise III (b)

1. Find the equation of a straight line.

   (*i*) parallel to $x$-axis and at a distance 7 units from it.

   (*ii*) parallel to $y$-axis and at a distance $-7$ units from it.

2. Find the equation of a straight line parallel to $x$-axis and passing through the point

   (*i*) (4, 7) (*ii*) (5, 11) (*iii*) $(h, k)$.

3. Find the equation of a straight line which

   (*i*) cuts OY at a distance 4 from O and makes an angle of 135° with OX.

   (*ii*) cuts OY at a distance $-6$ from O and makes an angle of 120° with OX.

(*iii*) passes through the point (2, 3) and makes an angle of 45° with OX.

4. Find the equation of a straight line which

(*i*) makes intercepts 5 and $-6$ on the axes.

(*ii*) passes through (4, 11) and makes equal intercepts on the axes. (*P.U.*)

(*iii*) passes through the point (7, 3) and makes intercepts on the axes which are equal in magnitude but opposite in sign.

5. Find the value of $m$ and $c$ so that the points (3, 7) and $(-2, 6)$ may lie on the line $y=mx+c$. (*P.U.*)

6. A straight line passes through the point (2, 3) and the portion of the line intercepted between the axes is bisected at the point ; find its equation. (*P.U.*)

7. Deduce from the equation $\frac{x}{a}+\frac{y}{b}=1$, the form $x \cos \alpha+y \sin \alpha=p$. (*P.U. 1943*)

8. Deduce from the equation $x \cos \alpha+y \sin \alpha=p$ the form $\frac{x}{a}+\frac{y}{b}=1$.

## 3.3. General Equation of the First Degree in x and y.

We have seen that the equations of the straight line that we have obtained so far are all of the first degree in $x$ and $y$. In the next article we propose to establish the converse theorem, namely that every equation of the first degree represents a straight line.

**3.31**. *Every equation of the first degree represents a straight line.*

The most general equation of the first degree is

$$Ax+By+C=0 \qquad \ldots(1)$$

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ be any three points on the locus, then the co-ordinates of these will satisfy the equation (1)

Hence $Ax_1+By_1+C=0 \qquad \ldots(2)$

$$Ax_2+By_2+C=0 \qquad \ldots(3)$$

$$Ax_3+By_3+C=0 \qquad \ldots(4)$$

Subtracting (3) from (2); we have

$$A(x_1-x_2)+B(y_1-y_2)=0$$

or $$A(x_1-x_2)=-B(y_1-y_2) \qquad \ldots(5)$$

Similarly subtracting (4) from (3), we have

$$A(x_2-x_3)+B(y_2-y_3)=0$$

or $$A(x_2-x_3)=-B(y_2-y_3) \qquad \ldots(6)$$

Dividing (5) by (6), we have

$$\frac{x_1-x_2}{x_2-x_3}=\frac{y_1-y_2}{y_2-y_3}$$

or $$(x_1-x_2)(y_2-y_3)=(x_2-x_3)(y_1-y_2)$$

or $$x_1y_2+x_2y_3+x_3y_1-y_1x_2-y_2x_3-y_3x_1=0$$

$\therefore$ P, Q and R lie on a straight line.

But P, Q and R are any three points on the locus represented by (1).

$\therefore$ locus is a straight line.

Hence $Ax+By+C=0$ represents a straight line.

**3.4.** In some of the previous articles, we have obtained the equation of a straight line satisfying different conditions in the following forms :

(*a*) $y=mx+c$, where $m$ is the slope of the line and $c$ is the intercept it cuts off from $y$-axes ;

(*b*) $\frac{x}{a}+\frac{y}{b}=1$, where $a$ and $b$ are the intercepts cut off by the straight line from the axes of $x$ and $y$ respectively.

(*c*) $x \cos \alpha+y \sin \alpha=p$, where $p$ is the length of the perpendicular from the origin on the straight line and $\alpha$ is the angle made by this perpendicular with the OX.

Since $Ax+By+C=0$ always represents a straight line it can be reduced to any of the above forms.

**3.41.** *To reduce the equation $AX+By+C=0$ to the form $y=mx+c$.*

The equation $Ax+By+C=0$ may be written as

$$By=-Ax-C$$

or $$y=-\frac{A}{B}X-\frac{C}{B}.$$

This is of the same form as $y=mx+c$, the slope being $-\frac{A}{B}$ and the intercept on $y$-axis $-\frac{C}{B}$.

**Example.** *Reduce the equation $2x-3y+7=0$ to the slope form and find its slope and the intercept on the $y$-axis.*

The equation can be written as

$$3y=2x+7$$

or $$y=\tfrac{2}{3}x+\tfrac{7}{3}$$ which is the slope form ;

the slope being $\frac{2}{3}$ and the intercept on $y$-axis $\frac{7}{3}$.

**3.42.** *To reduce the equation $AX+BY+C=0$ to the intercept form.*

We find that in the form $\frac{x}{a}+\frac{y}{b}=1$, the right-hand side is unity and left-hand side is the sum of two terms.

Now $Ax+By+C=0$, can be written as

$$Ax+By=-C$$

or $$\frac{A}{-C}x+\frac{B}{-C}y=1$$

$$\frac{x}{\frac{-C}{A}}+\frac{y}{\frac{-C}{B}}=1$$

which is the intercept form, the intercepts on the co-ordinate axes being $\frac{-C}{A}$ and $\frac{-C}{B}$.

**Example.** *Transform the equation $3x+4y+9=0$ to the intercept form.*

Now $3x+4y+9=0$ can be written as

$$3x+4y=-9$$

or $$\frac{3}{-9}x+\frac{4}{-9}y=1$$

or $\dfrac{x}{-3}+\dfrac{y}{\frac{-9}{4}}=1$, which is the intercept form, the intercepts on the axes being $-3$ and $\dfrac{-9}{4}$.

**3.4.3.** *Reduce the equation $Ax+By+C=0$, to the perpendicular form.*

The equation of the straight line in perpendicular form is

$$x \cos \alpha+y \sin \alpha=p$$

or $$x \cos \alpha+y \sin \alpha-p=0 \qquad \text{...(1)}$$

If the equation

$$\text{AX}+\text{BY}+\text{C}=0 \qquad \text{...(2)}$$

also represents the same straight line

$$\frac{\cos \alpha}{\text{A}}=\frac{\sin \alpha}{\text{B}}=\frac{-p}{\text{C}}$$

or $$\frac{-p}{\text{C}}=\frac{\cos \alpha}{\text{A}}=\frac{\sin \alpha}{\text{B}}=\frac{\sqrt{\cos^2 \alpha+\sin^2 \alpha}}{\pm\sqrt{\text{A}^2+\text{B}^2}}$$

$$=\frac{1}{\pm\sqrt{\text{A}^2+\text{B}^2}}.$$

$\therefore$ $$\cos \alpha=\frac{\text{A}}{\pm\sqrt{\text{A}^2+\text{B}^2}}, \qquad \sin \alpha=\frac{\text{B}}{\pm\sqrt{\text{A}^2+\text{B}^2}}$$

and $$p=\frac{-\text{C}}{\pm\sqrt{\text{A}^2+\text{B}^2}}$$

**Since $p$ is always positive, the sign before the radical must be that which makes $p$ positive.**

Hence if C is positive $p=\dfrac{-\text{C}}{-\sqrt{\text{A}^2+\text{B}^2}}=\dfrac{\text{C}}{\sqrt{\text{A}^2+\text{B}^2}}$

and, therefore, $\cos \alpha=\dfrac{\text{A}}{-\sqrt{\text{A}^2+\text{B}^2}}$

and $$\sin \alpha=\frac{\text{B}}{-\sqrt{\text{A}^2+\text{B}^2}}$$

$\therefore$ The given equation reduced to perpendicular form becomes

$$\frac{-A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}} \qquad \ldots(3)$$

If C is negative $p = \dfrac{-C}{\sqrt{A^2+B^2}}$

and, therefore $\cos\alpha = \dfrac{A}{\sqrt{A^2+B^2}}$

and $\sin\alpha = \dfrac{B}{\sqrt{A^2+B^2}}$.

Hence the given equation reduced to the perpendicular form becomes

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = \frac{-C}{\sqrt{A^2+B^2}} \qquad \ldots(4)$$

**Example.** *Transform the equations*

$3x+4y-7=0$

$12x+5y+4=0$, *to the perpendicular form.*

(*i*) $3x+4y-7=0$

Here C is negative, $\therefore$ dividing by $+\sqrt{A^2+B^2}$ *i.e.* 5 and transposing the constant term, we have

$\frac{3}{5}x+\frac{4}{5}y=\frac{7}{5}$ as the required equation.

Now $\cos\alpha=\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$ and $p=\frac{7}{5}$.

(*ii*) $12x+5y+4=0$

Here C is positive, $\therefore$ dividing by $-\sqrt{A^2+B^2}$ *i.e.* $-13$ and transposing the constant term, we have

$\dfrac{-12}{13}x-\dfrac{5}{13}y=\dfrac{4}{13}$ as the required equation.

Now $\cos\alpha=\dfrac{-12}{13}$, $\sin\alpha=\dfrac{-5}{13}$ and $p=\dfrac{4}{13}$

## Exercise III (c)

1. Reduce the following equations to the slope form :

(*i*) $2x-3y+2=0$ (*ii*) $4x+5y-1=0$

(*iii*) $x \cos \alpha + y \sin \alpha = p$ (*iv*) $\frac{x}{a} + \frac{x}{b} = 1$

2. Reduce the following equations to intercept form :

(*i*) $3x - 4y + 24 = 0$ (*ii*) $5x - 3y - 5 = 0$

(*iii*) $y = mn + c$ (*iv*) $x \cos \alpha + y \sin \alpha = p$.

3. Reduce the following equations to perpendicular form :

(*i*) $3x + 4y - 5 = 0$ (*ii*) $x + \sqrt{3}\, y + 7 = 0$

(*iii*) $y = mx + c$ (*iv*) $12x - 5y + 9 = 0$

4. Prove that the line $y - x + 2 = 0$ cuts the line joining (3, —1) and (8, 9) in the ratio 2 : 3.

[Let the ratio be 1 : $k$. $\therefore$ the coordinates of the point of division are $\left(\frac{+3k}{1+k}, \frac{9-k}{1+k}\right)$.

But this point lies on the line $x - y + 2 = 0$

$$\therefore \frac{9-k}{1+k} - \frac{8+3k}{1+k} + 2 = 0 \quad \text{or} \quad 3 - 2k = 0 \quad \therefore k = \tfrac{3}{2}]$$

5. In what ratio is the line joinining (1, 1) and (3, 4) divided by $2x - 3y - 7 = 0$ ?

6. In what ratio is the line joining (1, 3) and (2, 7) divided by $3x + y = 9$,

## Other Forms of the Equation of a Straight Line

**3.5.** *To find the equation of a straight line passing through a given point and having a given slope.*

Let the slope of the straight line be $m$ and let it pass through $(x_1, y_1)$.

The equation of a line with slope $m$ can be written as

$$y = mx + c \qquad \ldots(1)$$

where $c$ is unknown.

Since the line passes through $(x_1, y_1)$, we have

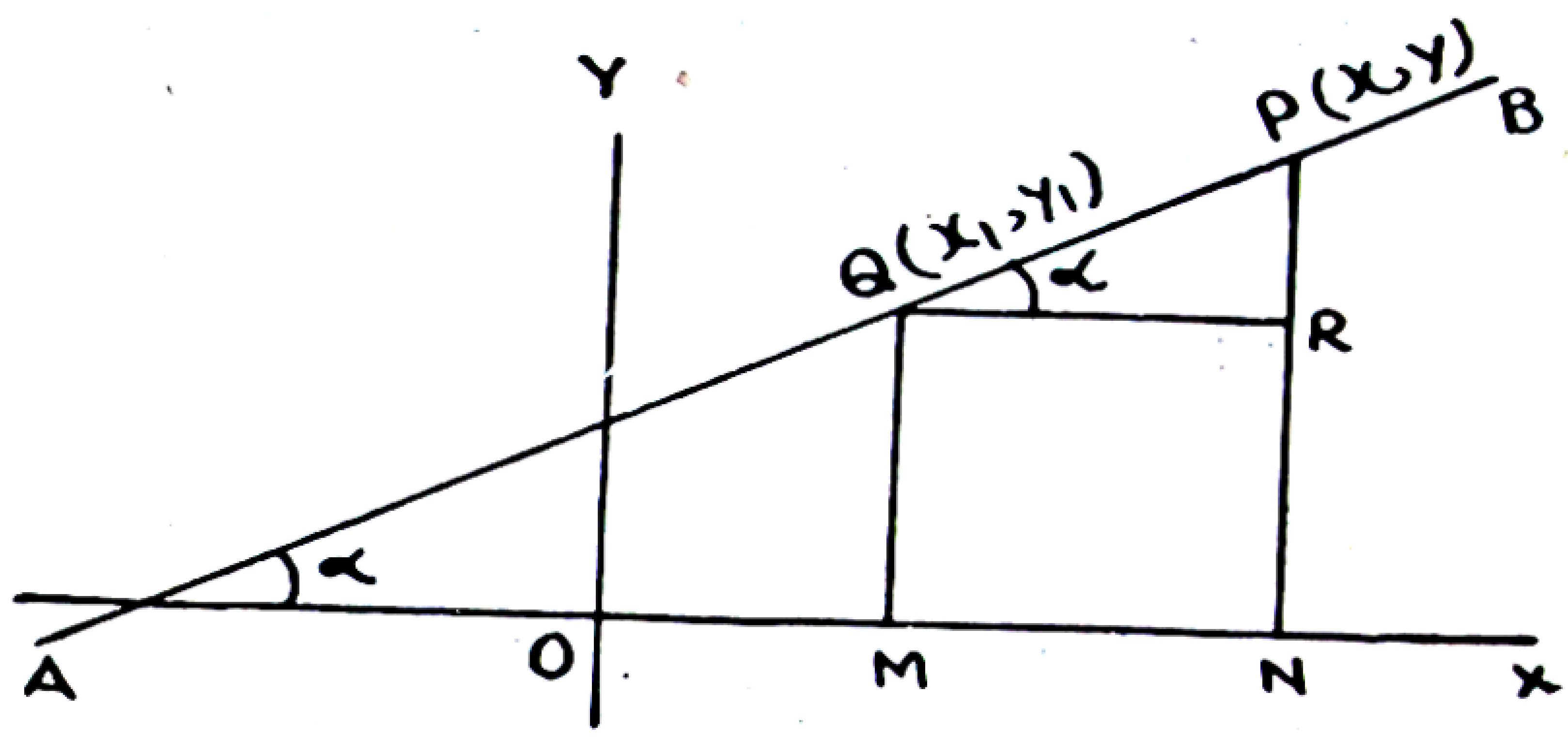
$$y_1 = mx_1 + c \qquad \ldots(2)$$

Subtracting (2) from (1), we have

$y - y_1 = m(x - x_1)$ as the required equation.

The above result can also be obtained in the following manner :

Let AB be the given linc passing through the point $Q(x_1, y_1)$. Let it make an angle $\alpha$ with OX such that $\tan \alpha = m$.




Take P $(x, y)$ any point on the line AB. Draw QM and PN perpendiculars to OX and QR perpendicular to NP meeting it in R.

Now $\angle PQR = \alpha$

$\therefore$ from the right angled triangle PQR,

$$\tan \alpha = \frac{RP}{QR} = \frac{y - y_1}{x - x_1}$$

$$\therefore \quad y = y_1 = \tan \alpha \ (x + x_1)$$

$$\text{or} \quad y - y_1 = m(x - x_1) \qquad \because \tan \alpha = m$$

**3.51.** *To find the equation of a straight line passing through two given points* $(x_1, y_1)$ *and* $(x_2, y_2)$.

Let the equation of the straight line be

$$y = mx + c$$

Since the points $(x_1, y_1)$ and $(x_2, y_2)$ lie on (1)

$$y_1 = mx_1 + c \qquad \ldots(2)$$

$$y_2 = mx_3 + c \qquad \ldots(3)$$

Subtracting (2) from (1) we get

$$y - y_1 = m(x - x_1) \qquad \ldots(4)$$

Sutracting (3) from (2) we get

$$y_1 - y_2 = m(x_1 - x_2) \qquad \ldots(5)$$

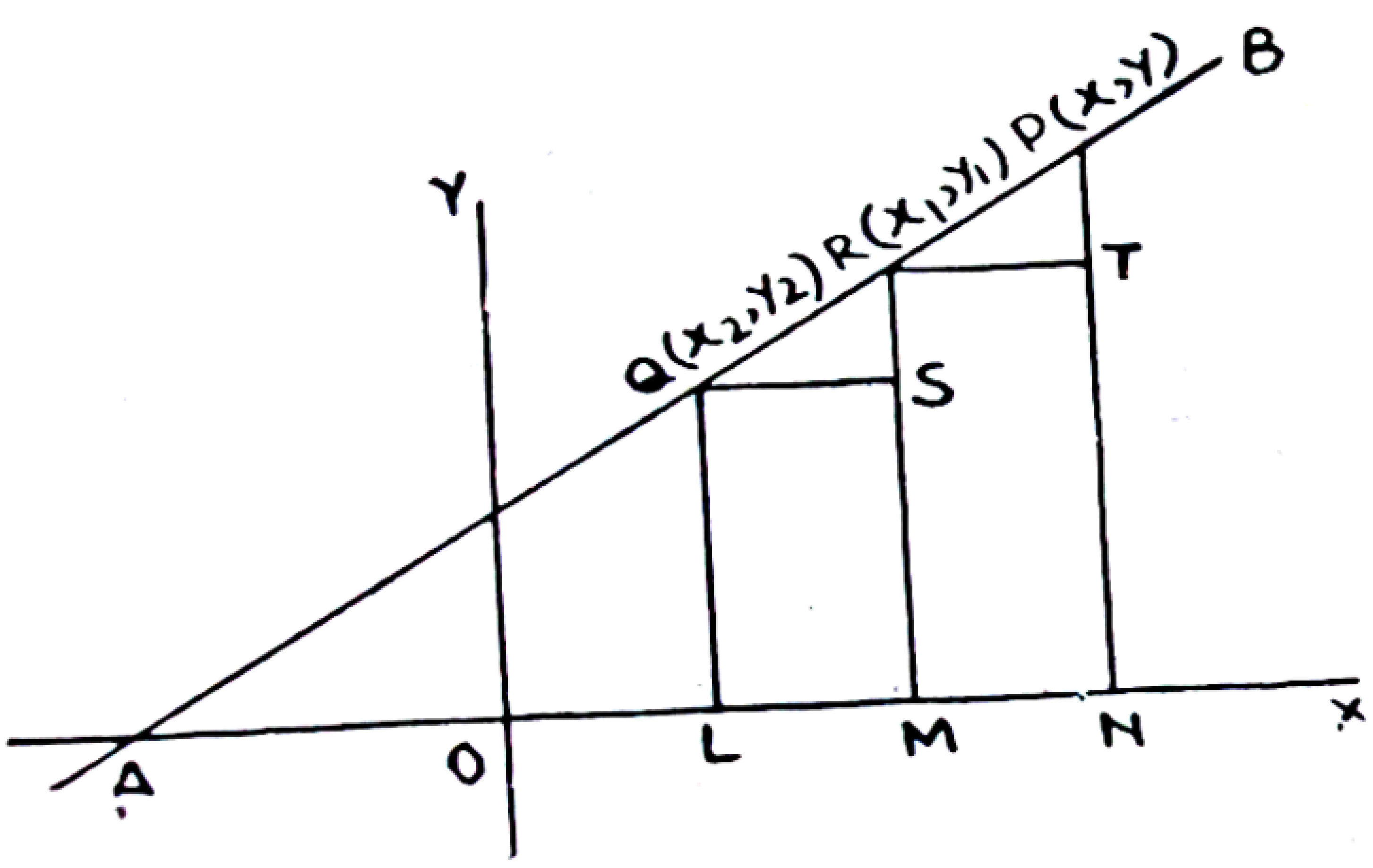
Dividing (4) by (5), we have

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}.$$

or $$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1)$$ is the required equation.

**Note.** $\frac{y_1-y_2}{x_1-x_2}$ is the slope of the line passing through $(x_1, y_1)$ and $(x_2, y_2)$.

The above result can also be obtained in the following manner :




Let AB be the given line passing through $Q(x_2, y_2)$ and $R(x_1, y_1)$.

Take $P(x, y)$ any point on the line.

Draw QL, RM, PN perpendiculars to OX and QS and RT perpendiculars to MR and NP respectively.

Now triangles PRT and RQS are similar.

$$\therefore \quad \frac{TP}{RT}=\frac{SR}{QS} \qquad \ldots(1)$$

But
$$TP=NP-NT=NP-MR=y-y_1$$
$$RT=MN \qquad =ON-OM=x-x_1$$
$$SR=MR-MS=MR-LQ=y_1-y_2$$
$$QS=LM \qquad =OM-OL=x_1-x_2$$

$\therefore$ from (1) we have

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

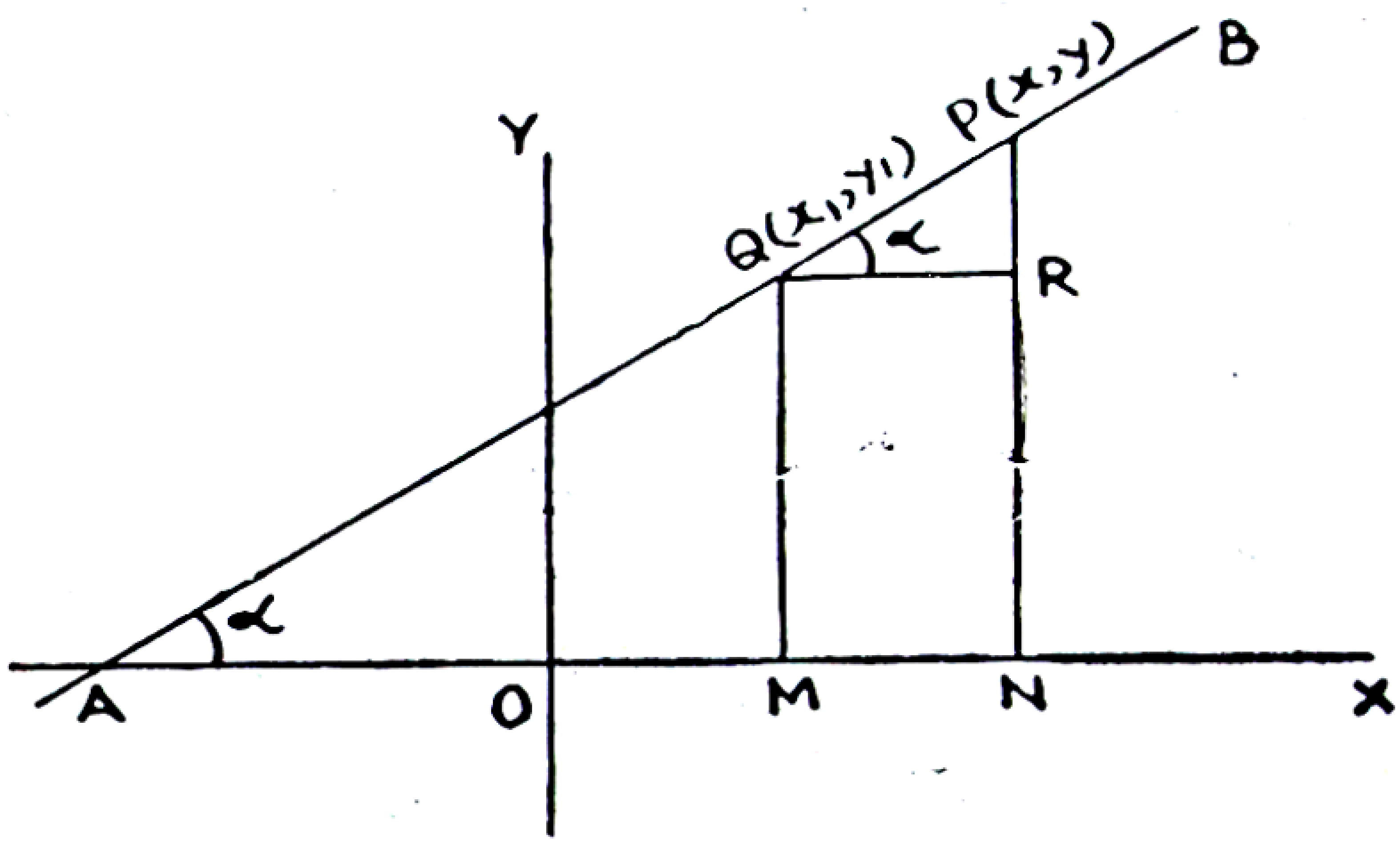
or $$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1)$$ is the required equation.

**3.52.** *To find the equation of a straight line passing through a given point $(x_1, y_1)$ and making a given angle $\alpha$ with the x-axis in the form*

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r.$$

*where $r$ is the algebraical distance of any point $(x, y)$ on the line from the point $(x_1, y_1)$.*

Let AB be the given line passing through the point Q $(x_1, y_1)$ and making an angle $\alpha$ with $x$-axis.




Let P $(x, y)$ be any point on the line such that

QP$=r$.

Draw QM and PN perpendiculars to OX and QR perpendicular to NP.

Now $\quad$ QR=MN=ON−OM$=x-x_1$

and $\quad$ RP=NP−NR=NP−MQ$=y-y_1$.

Also from the right-angled triangle PQR

$$QR=QP\cos\alpha$$

or
$$x-x_1=r\cos\alpha$$

$$\therefore \quad \frac{x-x_1}{\cos\alpha}=r \qquad \ldots(1)$$

and
$$RP=QP\sin\alpha$$

or
$$y-y_1=r\sin\alpha$$

$$\therefore \quad \frac{y-y_1}{\sin\alpha}=r \qquad \ldots(2)$$

Hence from (1) and (2) we have

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r$$

**Cor.** From (1) and (2) we have

$$x=x_1+r\cos\alpha$$

and
$$y=y_1+r\sin\alpha$$

These equations give us co-ordinates of any point on the straight line in terms of $r$.

*These equations are therefore called the parametric equations of a straight line.*

**Example 1.** *Find the equation of the straight line passing through the points (3, 5) and (−2, 1).*

Now $x_1=3,\ y_1=5,\ x_2=-2$ and $y_2=1$

$\therefore$ the required equation is

$$\frac{y-5}{x-3}=\frac{5-1}{3+2}$$

or
$$\frac{y-5}{x-3}=\frac{4}{5}$$

or
$$5y-25=4x-12$$

or
$$4x-5y+13=0.$$

**Example 2.** *Find the distance of the point (3, 4) from the line $2x-y-7=0$ measured along a line making an angle of 45° with x-axis.*

The equation of a line passing through (3, 4) and making an angle of 45° with $x$-axis is

$$\frac{x-3}{\cos 45}=\frac{y-4}{\sin 45}=r.$$

$\therefore$ the co-ordinates of any point on the line are

$$\left(3+\frac{r}{\sqrt{2}},\ 4+\frac{r}{\sqrt{2}}\right).$$

The value of $r$ for which this point satisfies the equation $2x-y-7=0$ is the required distance.

Now
$$6+\frac{2r}{\sqrt{2}}-4-\frac{r}{\sqrt{2}}-7=0$$

or
$$\frac{r}{\sqrt{2}}-5=0$$

$$r=5\sqrt{2}.$$

**Exercise III (d)**

Find the equations to the straight lines passing through the following pairs of points :—

1. (0, 0) and (2, −2).

2. $(-1, 3)$ and $(6, -7)$.

3. $(3, 5)$ and $(-2, 1)$.

4. $(at_1^2, 2at_1)$, and $(at_2^2, 2at_2)$.

5. $(a \cos \phi, a \sin \phi)$ and $(a \cos \theta, a \sin \theta)$.

6. $(a \cos \phi, b \sin \phi)$ and $(a \cos \theta, b \sin \theta)$.

7. Find the equations to the sides of a triangle the co-ordinates of whose angular points are

$$(1, 4), (2, -3) \text{ and } (-1, -2).$$

8. Prove that the line joining $(3, 5)$ and $(-2, 7)$ bisects the line joining $(7, 2)$ and $(9, 4)$.

9. Prove that the line through the two points $(9, 3)$ and $(15, -3)$ cuts off equal intercepts from the axes.

10. In what ratio is the line joining the points $(1, 2)$ and $(4, 3)$ divided by the line joining $(2, 3)$ and $(4, 1)$.

11. Find the co-ordinates of the points which are at a distance of 5 units from the point $(3, 4)$ and lie on the lines $4x-3y=0$.

12. Prove that the three points $(-1, -1)$, $(5, 7)$ and $(8, 11)$ lie in a straight line. Find the intercepts which it makes on the axes. (*P.U. 1945*)

13. Interpret completely the equations

(*i*) $x=a$ (*ii*) $y=b$ (*iii*) $y=mx$. (*P.U. 1944*)

14. Find the equation of the line which makes equal intercepts on the axes and passes through the point $(2, 3)$. (*E.P.U. 1948*)

15. The part between the axes of a line passing through $(a, b)$ is bisected at that point. Find the equation of the line.

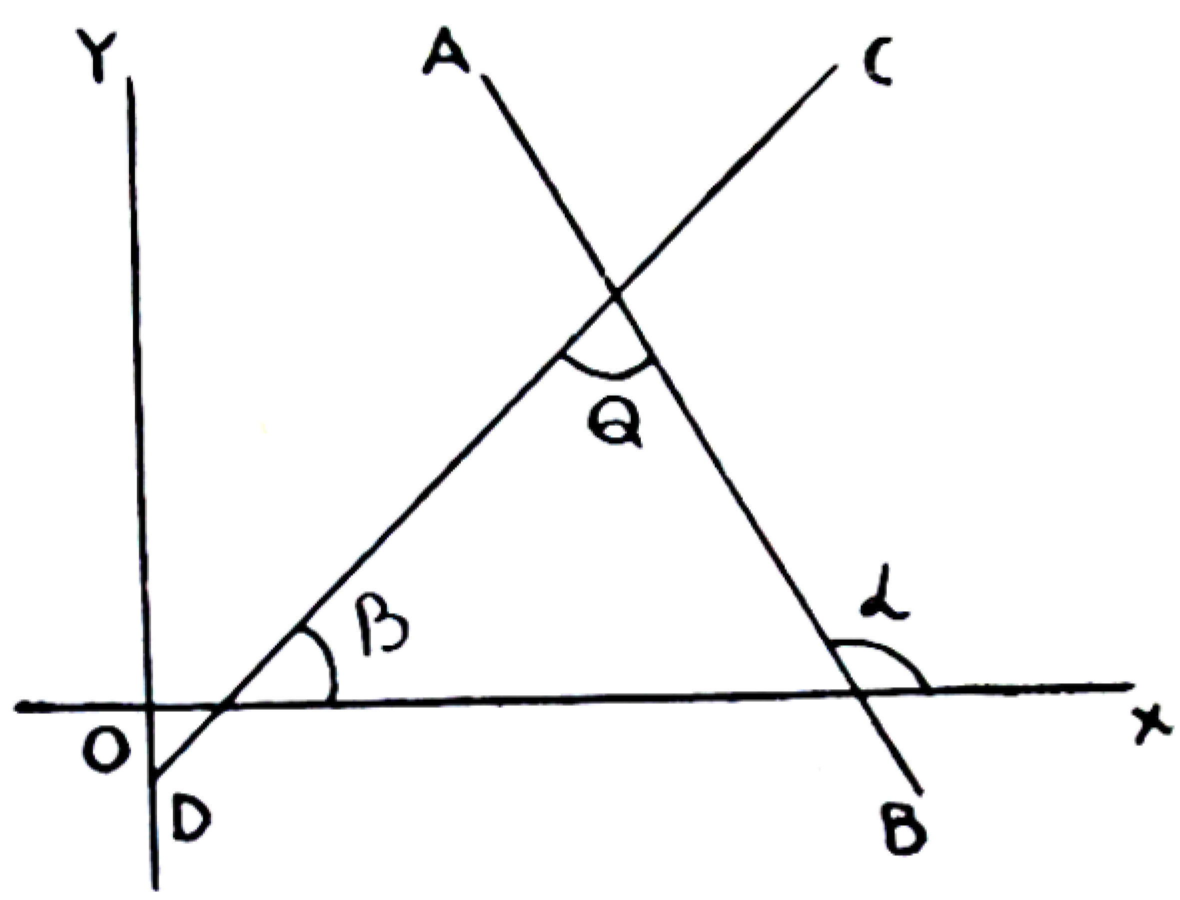
16. A rectangle is bounded by the co-ordinate axes and the lines $x=6$, $y=8$. Find the equations of the diagonals.

17. Find the distance of the point $(2, 3)$ from the line $x+y+1=0$, measured along a line making an angle of 45° with $x$-axis.

CHAPTER IV

# THE STRAIGHT LINE—(*Contd.*)

## ✓ 4.1. Angle between two straight lines.




Let AB and AC be the two given straight lines with inclinations $\alpha$, $\beta$ and their equations in slope form be $y=m_1x+c_1$ and $y=m_2x+c_2$ respectively

Then $\tan\alpha=m_1$

and $\tan\beta=m_2$

Let $\theta$ be the angle between the lines.

Then $\theta=\alpha-\beta$

$\therefore \quad \tan\theta=\tan(\alpha-\beta)$

$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{m_1-m_2}{1+m_1m_2}$$

or $$\theta=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$$

**Note.** 1. The positive value of $\dfrac{m_1-m_2}{1+m_1m_2}$ gives the acute angle while the negative value, the obtuse angle.

2. If $m_1=m_2$, $\tan\theta=0$, or $\theta=0$

$\therefore$ the two lines are parallel.

3. If the $m_1m_2=-1$, $\tan\theta\to\infty$, $\theta=\dfrac{\pi}{2}$

$\therefore$ the two lines are perpendicular to each other.

**4.11.** If the equations of the two lines are

$$a_1x+b_1y+c_1=0$$

and $$a_2x+b_2y+c_2=0$$

these can be written as

$$y=-\frac{a_1}{b_1}x-c_1$$

and $$y=-\frac{a_2}{b_2}x-c_2$$

so that $$m_1=\frac{-a_1}{b_1}$$

$$m_2=-\frac{a_2}{b_2}$$

Now $$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

$$=\frac{\dfrac{-a_1}{b_1}+\dfrac{a_2}{b_2}}{1+\dfrac{a_1}{b_1}\cdot\dfrac{a_2}{b_2}}$$

$$=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

or $$\theta=\tan^{-1}\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

The two lines are parallel if $a_2b_1-a_1b_2=0$

*i.e.* if $a_1b_2=a_2b_1$ or $\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}$

the lines are perpendicular to each other if

$$a_1a_2+b_1b_2=0.$$

**4.12.** *To find the equation of a straight line parallel to a given line.*

Let the equation of the given line be

$$ax+by+c=0.$$

The slope of this line is $\dfrac{-a}{b}$.

The slope of the ∥ line is $\frac{-a}{b}$.

Hence the required equation is

$$y=-\frac{a}{b}x+c_1$$

or $\quad ax+by+k=0 \qquad$ where $\qquad k=-bc_1$

Hence the equations of ∥ lines differ only in the constant term.

**4.13.** *To find the equation of a straight line perpendicular to a given line.*

Let $a_1x+b_1y+c_1=0$ be the given line.

Now $a_2x+b_2y+c_2=0$ will be perpendicular to it if

$$a_1a_2+b_1b_2=0$$

or $$\frac{a_2}{b_1}=-\frac{b_2}{a_1}=l \qquad \text{(say)}$$

$\therefore \qquad a_2=b_1l \qquad$ and $\qquad b_2=-a_1l$

Substituting these values of $a_2$ and $b_2$ we get

$$lb_1x-la_1y+c_2=0$$

or $$b_1x-a_1y+k=0$$

where $$\frac{c_2}{l}=k$$

which is the required line.

**Rule** :—*Hence to find the equation of a line perpendicular to a given line*

(*i*) *interchange the coefficients of $x$ and $y$ and change the sign of one of them.*

*and* (*ii*) *change the constant term into any other constant.*

**Ex. 1.** *Find the angle between the lines*

$$2x-y+5=0 \qquad \text{and} \qquad 2x+3y-7=0$$

**Here** $\quad m_1=2 \qquad$ and $\qquad m_2=\frac{-2}{3}$

$\therefore$ If $\theta$ is the angle between the lines

$$\tan\ \theta = \frac{2-\left(\frac{-2}{3}\right)}{1+2\times\left(\frac{-2}{3}\right)} = \frac{2+\frac{2}{3}}{1-\frac{4}{3}} = -8$$

$\therefore$ the acute angle is $\tan^{-1} 8$.

**Ex. 2.** *Find the equation of the straight line through the point $(-2, -1)$ and parallel to the line $3x+4y+7=0$.* (1942)

The equation of any line parallel to $3x+4y+7=0$ is given by

$$3x+4y+k=0$$

The line passes through the point $(-2, -1)$

$\therefore$ $-6-4+k=0$ $\therefore$ $k=10$

$\therefore$ the required line is

$$3x+4y+10=0.$$

**Ex. 3.** *Find the equation of a straight line through the point $(4, 5)$, perpendicular to the line $3x-2y+5=0$.*

Any line perpendicular to the given line is given by

$$2x+3y+k=0.$$

The line passes through the point $(4, 5)$ if

$$8+15+k=0 \quad \text{or} \quad k=-23.$$

Substituting the value of $k$, we get

$$2x+3y-23=0$$

which is the required equation.

**Ex. 4.** *Find the equations of the straight lines passing through the point $(3, -2)$ and inclined at an angle of $60°$ to the line* $\sqrt{3}.x+y=1$. (1944)

Let $m$ be the slope of the required line.

Slope of the given line $=-\sqrt{3}$.

Since the angle between the required line and the given line is $60°$, hence

$$\tan 60° = \pm \frac{m-(-\sqrt{3})}{1-\sqrt{3}.m}$$

or $$\sqrt{3}=\pm\frac{m+\sqrt{3}}{1-\sqrt{3}m}$$

Taking the $+ve$ sign,

$$\sqrt{3}-3m=m+\sqrt{3} \quad \text{or} \quad m=0$$

Taking the $-ve$ sign,

$$-\sqrt{3}+3m=m+\sqrt{3} \quad \text{or} \quad m=\sqrt{3}$$

As the lines pass through $(3, -2)$,

the required equations are

$$y+2=0$$

and $$y+2=\sqrt{3}\,(x-3)$$

*i.e.* $$\sqrt{3}.x-y=2+3\sqrt{3}.$$

## Exercise IV (a)

1. Find the angle between the straight lines

 (*i*) $y\sqrt{3}-x+4=0$ and $x+\sqrt{3}y-3=0$.

 (*ii*) $ax+by+c=0$ and $(a+b)x-(a-b)y=0$.

 (*iii*) $x\cos\alpha+y\sin\alpha=p$ and $x\cos\beta+y\sin\beta=p'$.

2. Find the angle between the two straight lines whose intercepts on the axes are $a$, $b$ and $a'$, $b'$ respectively.

3. Show that the lines $y-2x=3$ and $2y=4x+5$ are parallel.

4. Find the equation of the straight line

 (*i*) passing through $(2, 3)$, and parallel to $3x-4y+5=0$. (*P.U.*)

 (*ii*) passing through the point $(1, 1)$ and parallel to the line $4x+4y+7=0$. (*1942*)

 (*iii*) passing through $(x_1, y_1)$ and parallel to $ax+by+c=0$.

5. Find the equation of the line

 (*i*) through the origin and perpendicular to $5x+12y+13=0$. (*P.U.*)

 (*ii*) through the point $(x_1, y_1)$ perpendicular to the line $Ax+By+c=0$. (*P.U.*)

6. Find the equation of the straight line through the point $(x^1, y^1)$ and perpendicular to $x^1y+xy^1=a^2$.

7. Obtain the equation of a line

(i) which passes through $(h, k)$ and which is perpendicular to $lx+my-1=0$ *(1945)*

(ii) which passes through the point $(a, b)$ and is at right angles to the line $px-qy-r=0$. *(1945S)*

8. Find the equation of the right bisector of the line joining the points $(3, -7)$ and $(-1, 5)$.

9. Find the equations of the altitudes of the triangle whose vertices are $(5, 2)$, $(-1, 1)$, $(2, 7)$.

10. Find the equations of the straight lines

(i) through the point $(4, 5)$ which make an acute angle of $45°$ with the line $2x-y+7=0$. *(1946)*

(ii) through the point $(h, k)$ making a given angle $\alpha$ with the straight line $y=mx+c$.

## Intersection of straight lines

### 4.2. Point of intersection of two given straight lines.

Let the equations of the two lines be

$$a_1x+b_1y+c_1=0 \qquad \ldots (1)$$
$$a_2x+b_2y+c_2=0 \qquad \ldots (2)$$

The point of intersection of two lines is common to both the lines. Hence its co-ordinates must satisfy both the equations. Thus we have only to solve these simultaneous equations to find $x$ and $y$.

By cross-multiplication, we have

$$\frac{x}{b_1c_2-c_1b_2}=\frac{y}{c_1a_2-a_1c_2}=\frac{1}{a_1b_2-a_2b_1}$$

$$i.e., \quad x=\frac{b_1c_2-c_1b_2}{a_1b_2-a_2b_1}, \quad y=\frac{c_1a_2-a_1c_2}{a_1b_2-a_2b_1}$$

This gives us the co-ordinates of the point of intersection.

**Note.** If $a_1b_2-a_2b_1$ is zero, there is no finite point of intersection. Or we say that the two lines meet at infinity, *i.e.*, they are parallel.

**Example 1.** *Find the point of intersection of the lines*

$$4x-6y=24 \quad \text{and} \quad 3x+7y+5=0$$

The two equations can be written as

$$4x-6y-24=0 \qquad \ldots(1)$$
$$3x+7y+5=0 \qquad \ldots(2)$$

By cross-multiplication

$$\frac{x}{-30+168}=\frac{y}{-72-20}=\frac{1}{28+18}$$

*i.e.*, $x=3$ and $y=-2$

$\therefore$ the point of intersection is $(3, -2)$.

**Example 2.** *Find the area of the triangle formed by the lines* $x+y=0$, $y=x+6$, and $y=7x+5$. *(1941)*

The equations of the three sides are

$$x+y=0 \qquad \ldots(1)$$
$$x-y+6=0 \qquad \ldots(2)$$
$$7x-y+5=0 \qquad \ldots(3)$$

Solving these equations simultaneously in pairs, we get $(-3, 3)$ as the point of intersection of (1) and (2), $(\frac{1}{6}, \frac{37}{6})$ that of (2) and (3) and $\left(\frac{-5}{8}, \frac{5}{8}\right)$ of (3) and (1).

These three points of intersection are the three vertices of the triangle whose area is to be calculated.

$\therefore$ the area of the triangle

$$=\tfrac{1}{2}[x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_3y_2-x_1y_3]$$
$$=\frac{1}{2}\left[\frac{-37}{2}+\frac{5}{48}-\frac{15}{8}-\frac{1}{2}+\frac{185}{48}+\frac{15}{8}\right]$$
$$=\frac{-361}{48}.$$

Neglecting the $-ve$ sign, the area is equal to $\frac{361}{48}$.

**4.21.** Consider the equation

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \quad \ldots(1)$$

where $k$ is a constant.

This equation represents a straight line, as it is of first degree. Also if there is a point $(x_1, y_1)$ which lies on each of the two straight lines given by

$$a_1x+b_1y+c_1=0 \qquad ...(2)$$
$$\text{and } a_2x+b_2y+c_2=0 \qquad ...(3)$$

it lies on the line given by (1) also, as $a_1x_1+b_1y_1+c_1=0$ and $a_2x_1+b_2y_1+c_2=0$.

This shows that equation (1) represents a straight line passing through the point of intersection of the lines (2) and (3), whatever value $k$ may have *i.e.*, equation (1) represents a system of straight lines, all of which pass through the point of intersection of the two lines.

**Example 1.** *Find the equation of the straight line passing through (3, 2) and the point of intersection of the lines* $2x-3y=12$ and $3x+7y+5=0$.

The equation of any lines passing through the point of intersection of the given lines is $2x-3y-12+k(3x+7y+5)=0$.

This line passes through (3, 2),

$\therefore \quad 6-6-12+k(9+14+5)=0 \quad$ or $\quad k=\frac{12}{28}=\frac{3}{7}$

$\therefore \quad$ the required equation is

$$2x-3y-12+\tfrac{3}{7}(3x+7y+5)=0$$

or $\quad 14x-21y-84+9x+21y+15=0$

*i.e.*, $\quad 23x-69=0 \qquad$ or $\qquad x-3=0$

This example can be done in another way also. The point of intersection of the given lines is $(3, -2)$. Now we find the equation of the line passing through $(3, -2)$ and $(3, 2)$.

The equation is $x=3$, as obtained before.

**Example 2.** *Find the equation of the line passing through the intersection of the lines* $3x-2y+1=0$ *and* $4x+2y+3=0$ *and parallel to* $5x-y+7=0$.

The equation of any line through the point of intersection of the given lines is

$$3x-2y+1+k(4x+2y+3)=0 \qquad ...(1)$$

The slope of (1) is $-\dfrac{3+4k}{-2+2k}$

But the line given by (1) is parallel to

$$5x-y+7=0 \qquad \ldots(2)$$

$$\therefore \qquad -\frac{3+4k}{-2+2k}=5 \quad i.e., \; k=\tfrac{1}{2}$$

$\therefore$ The reqd. equation is $3x-2y+1+\frac{1}{2}(4x+2y+3)=0$

or $\qquad 10x-2y+5=0.$

**Example 3.** *Find the equation of the line through the intersection of the lines $3x+4y=7$ and $4x-5y+11=0$ and perpendicular to $3x-2y=1$.*

Equation of any line through the intersection of the given line is

$$3x+4y-7+k(4x-5y+11)=0$$

Slope of this line $=-\dfrac{3+4k}{4-5k}$

Slope of the line $(3x-2y=1)=\frac{3}{2}$

The two lines are perpendicular

$$\therefore \qquad -\frac{3+4k}{4-5k}\times\frac{3}{2}=-1$$

or $\qquad 9+12k=8-10k$

or $\qquad k=-\frac{1}{22}$

$\therefore$ The reqd. equation is

$$3x+4y-7-\tfrac{1}{22}(4x-5y+11)=0$$

*i.e.,* $\qquad 62x+93y-165=0.$

## 4.22. Condition of concurrency of three lines.

Let the equations of the three lines be

$$a_1x+b_1y+c_1=0 \qquad \ldots(1)$$

$$a_2x+b_2y+c_2=0 \qquad \ldots(2)$$

$$a_3x+b_3y+c_3=0 \qquad \ldots(3)$$

The co-ordinates of the point of intersection of (1) and (2), obtained by solving the two equations simultaneously, are

$$\left[\frac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2},\; \frac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}\right]$$

The three lines will be concurrent if this point lies on the line (3) *i.e.*, if

$$a_3 \times \frac{b_1c_2-c_1b_2}{a_1b_2-b_1a_2}+b_3 \times \frac{c_1a_2-a_1c_2}{a_1b_2-b_1a_2}+c_3=0$$

or $\quad a_3(b_1c_2-c_1b_2)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)=0$

which is the required condition.

**Otherwise :** If $l$, $m$, $n$ three non-zero constants can be found so that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)\equiv 0 \quad \ldots(i)$$

then the three straight lines are concurrent.

For if $(x_1, y_1)$ be the co-ordinates of the point of intersection of the first two of the lines, we have

$$a_1x_1+b_1y_1+c_1=0$$
$$a_2x_1+b_2y_1+c_2=0$$

and therefore on account of relation (*i*)

$$a_3x_1+b_3y_1+c_3=0$$

This shows that $(x_1\ y_1)$ lies on the third line also. Hence the three lines are concurrent.

**Example.** *Prove that the right bisectors of the sides of a triangle are concurrent.*

Let $A(x_1\ y_1)$, $B(x_2\ y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle.

The right bisector of the side BC is

$$(x-x_2)^2+(y-y_2)^2=(x-x_3)^2+(y-y_3)^2$$

or $\quad 2x(x_3-x_2)+2y(y_3-y_2)+x_2^2-x_3^2+y_2^2-y_3^2=0 \quad \ldots(i)$

Similarly other two right bisectors are

$$2x(x_1-x_3)+2y((y_1-y_3)+x_3^2-x_1^2+y_3^2-y_1^2=0 \quad \ldots(ii)$$

and $2x(x_2-x_1)+2y(y_2-y_1)+x_1^2-x_2^2+y_1^2-y_2^2=0 \quad \ldots(iii)$

Adding (*i*), (*ii*) and (*iii*) we find that they vanish identically.

Hence the right bisectors are concurrent.

## Exercise IV (b)

1. Find the point of intersection of the straight lines

(i) $4x+3y=10$ and $3x+5y=13$

(ii) $3x-2y+1=0$ and $2x+5y-31=0$

(iii) $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$.

2. Find the area of the triangle formed by the lines

(i) $3x+y+4=0$, $5x-5y+34=0$, $3x-2y+1=0$

(ii) $y-x=0$, $y+x=0$, $x-c=0$.

3. Find the equation of the straight line :—

(i) joining the origin to the point of intersection of $4x+3y=8$ and $x+y=1$

(ii) joining the point (2, 3) to the intersection of $2x+3y+1=0$ and $3x-4y=5$

(iii) through the intersection of $3x+2y=8$ and $5x-11y+1=0$ and parallel to $6x+13y=25$ (*1937S.*)

(iv) joining the point $(x', y')$ to the point of intersection of the lines $ax+by+c=0$ and $a'x+b'y+c'=0$ (*1938*)

(v) through the intersection of the lines $3x+4y=7$ and $4x-5y+11=0$ and perpendicular to $3x-2y=1$.

4. Find the equations of the altitudes and co-ordinates of the ortho centre of the triangle whose vertices are (1, 0), (2, −4), (−5, −2). (*1943 S.*)

5. Prove that the following lines are concurrent :

(i) $x+3y+5=0$, $4x+6y-1=0$, $3x+5y+1=0$

(ii) $2x-y=5$, $3x-y=6$, $4x-y=7$.

6. Find the condition that the following lines may be concurrent :

(i) $y=m_1x+c_1$, $y=m_2x+c_2$, $y=m_3x+c_3$.

7. Find the value of $a$ so that the line $x-6y+a=0$ may pass through the intersection of the lines

$$2x+3y+4=0 \text{ and } x+4y+1=0.$$

8. Prove that the

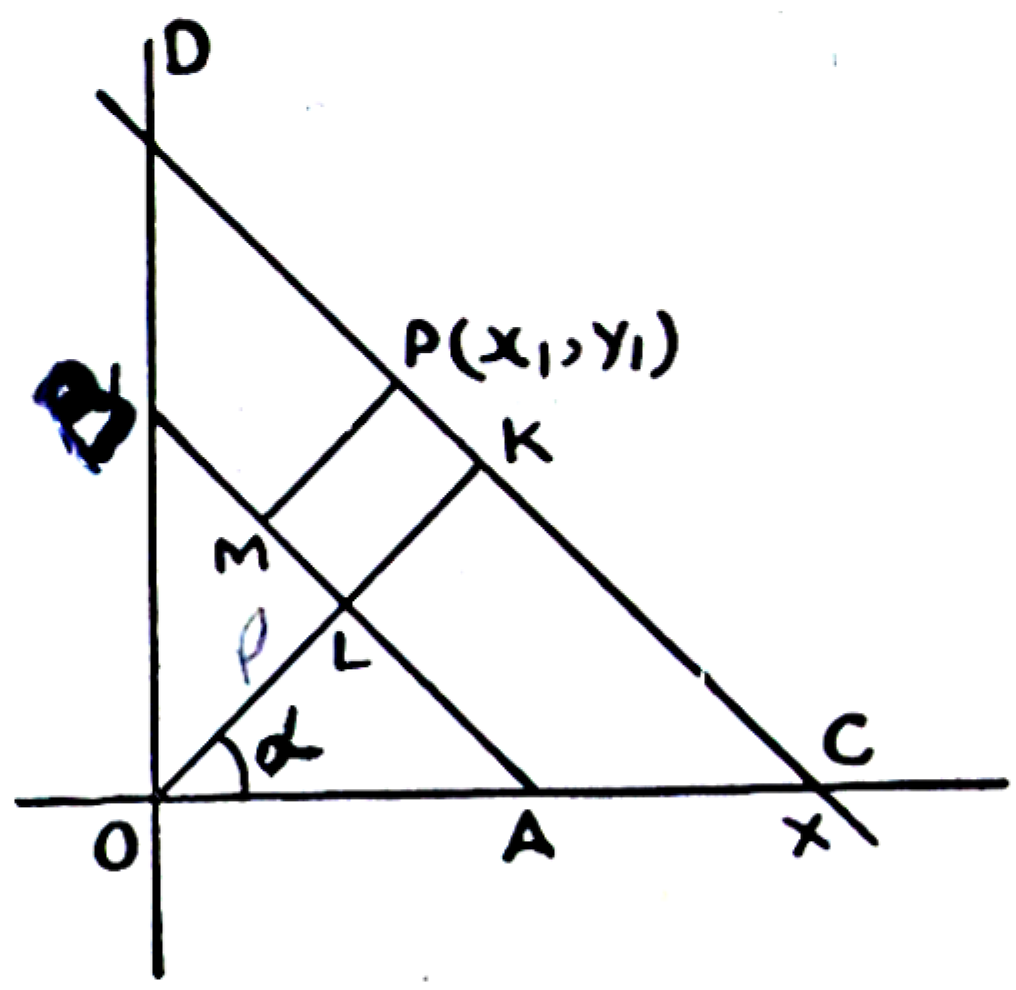
(*i*) medians of a triangle are concurrent,

(*ii*) altitudes of a triangle are concurrent.

## Distance of a point from a straight line

**4.3. Perpendicular distance of the point $(x_1, y_1)$ from the straight line $x \cos \alpha+y \sin \alpha=p$.**

Let AB be the given line. Let OL be perpendicular on it from O. Then $OL=p$ and $\angle LOA=\alpha$. Let $P(x_1, y_1)$ be the given point.




Through P draw a line CD∥AB. Produce OL to meet CD in K. From P draw PM perpendicular to AB. Let $OK=p_1$.

Now as OK is perpendicular to CD which is parallel to AB, the equation of CD is

$$x \cos \alpha+y \sin \alpha=p_1 \quad \ldots(1)$$

The point $P(x_1, y_1)$ lies on (1),

$$\therefore \quad x_1 \cos \alpha+y_1 \sin \alpha=p_1 \quad \ldots(2)$$

Perpendicular distance of P from AB is PM and

$$PM=KL=KO-LO=p_1-p$$
$$=x_1 \cos \alpha+y_1 \sin \alpha-p.$$

**Thus it follows that when the equation of a straight line is given in the form**

$$x \cos \alpha+y \sin \alpha-p=0$$

**if the co-ordinates of any point are substituted for x and y in the expression on the left-hand side of the equation the result represents numerically the distance of the point from the line.**

**4.31. Perpendicular distance of the point $(x_1, y_1)$ from the straight line $ax+by+c=0$.**

Let the equation

$$ax+by+c=0$$

be so written that $c$ is a negative quantity. This equation is reduced to perpendicular form by dividing it by $\sqrt{a^2+b^2}$. It therefore becomes

$$\frac{ax}{\sqrt{a^2+b^2}}+\frac{by}{\sqrt{a^2+b^2}}+\frac{c}{\sqrt{a^2+b^2}}=0$$

where $\cos\alpha=\dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha=\dfrac{b}{\sqrt{a^2+b^2}}$, $-p=\dfrac{c}{\sqrt{a^2+b^2}}$.

The perpendicular distance from $(x', y')$

$$=x'\cos\alpha+y'\sin\alpha-p$$

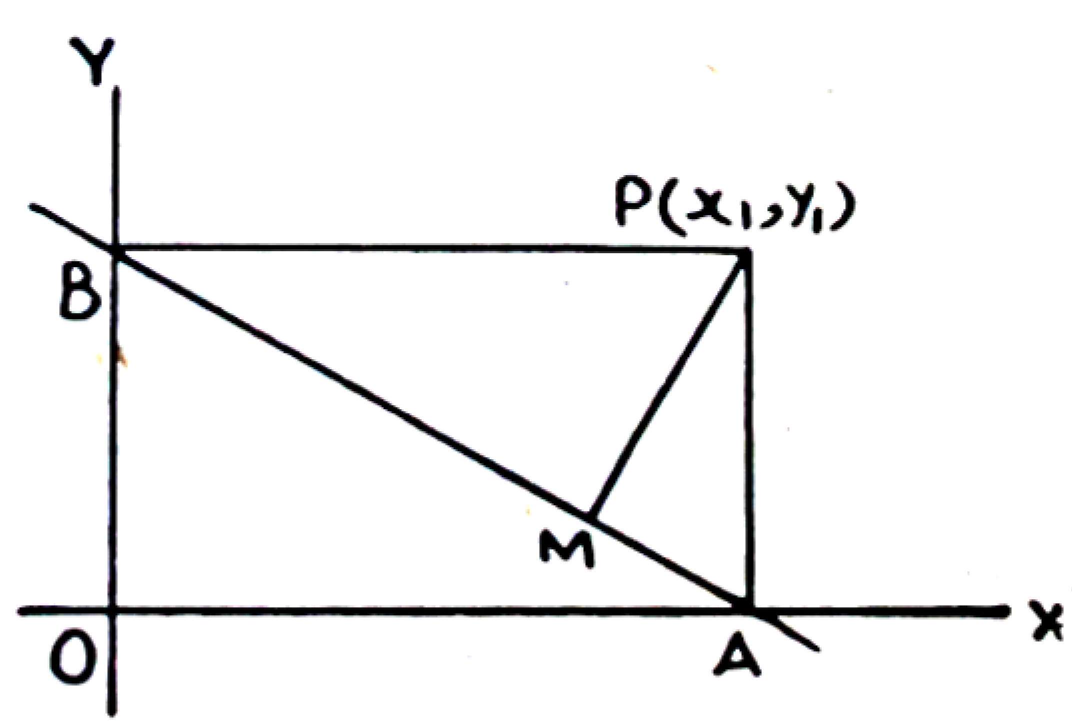
$$=\frac{ax'+by'+c}{\sqrt{a^2+b^2}}$$

**Thus the length of the perpendicular from $(x', y')$ on $ax+by+c=0$ is obtained by substituting $x'$ and $y'$ for $x$ and $y$ in the left-hand side of the same, and dividing the result so obtained by the square root of the sum of the squares of the coefficients of $x$ and $y$.**

**Cor. 1.** The length of the perpendicular from the origin on the line $ax+by+c=0$ is $\dfrac{c}{\sqrt{a^2+b^2}}$.

**4.2.** The result of the preceding article can also be obtained independently as follows :—

Let AB be the line whose equation is $ax+by+c=0$. Let $P(x_1, y_1)$ be the given point. Draw PM perpendicular to AB. Join PA and PB.




The co-ordinates of the point A, where AB meets the $x$-axis are obtained by solving

$$ax+by+c=0 \qquad \ldots(1)$$

and $y=0$ simultaneously.

Putting $y=0$ in (1), we get

$$ax+c=0 \qquad \text{or} \quad x=\frac{-c}{a}$$

$\therefore$ A is $\left(\frac{-c}{a}, 0\right)$.

Similarly B is $\left(0, \frac{-c}{b}\right)$

Now area of the $\triangle APB=\frac{1}{2}AB.MP=\frac{1}{2}AB\cdot p$

$$\text{or} \quad p=\frac{2\cdot\triangle APB}{AB} \qquad \ldots(2)$$

The vertices of the $\triangle$ APB are

$$\left(\frac{-c}{a}, 0\right), \left(0, \frac{-c}{b}\right), (x_1, y_1).$$

$$\therefore \quad \text{its area}=\frac{1}{2}\left[\frac{c^2}{ab}+\frac{cx_1}{b}+\frac{cy_1}{a}\right]$$

$$=\frac{c}{2ab}(ax_1+by_1+c)$$

Also $\quad AB=\sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}}=\frac{c}{ab}\sqrt{a^2+b^2}$

$\therefore$ from (2) we get

$$p=2.\frac{c}{2ab}(ax_1+by_1+c)\times\frac{ab}{c\sqrt{a^2+b^2}}$$

$$=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

**Example 1.** *Find the length and the equation of the perpendicular from the point (1, —2) on the line 3y=4x—5. (1949)*

The equation of the line can be written as

$$4x-3y-5=0 \qquad \ldots(1)$$

$\therefore$ The length of the perpendicular from (1, —2)

$$=\frac{4\cdot1-3(-2)-5}{\sqrt{4^2+3^2}}=\frac{5}{5}=1$$

The equation of any line perpendicular to (1) is

$$3x+4y+k=0 \qquad \ldots(2)$$

As the point (1, $-2$) lies on (2)

$\therefore$ $3-8+k=0$ or $k=5$.

$\therefore$ the equation of the required perpendicular is

$$3x+4y+5=0 \qquad \ldots(3)$$

**Note.**—The foot of the perpendicular from (1, $-2$) on the given line can be found by solving the equations (1) and (3) simultaneously.

**Example 2.** *Find the foot of the perpendicular from (3, 13) on the line $2x+3y-32=0$.*

Equation of the line passing through (3, 13) and perpendicular to $2x+3y-32=0$ is

or $3(x-3)-2(y-13)=0$

$$3x-2y+17=0.$$

The foot of perpendicular is the point of intersection of

$$2x+3y-32=0 \qquad \ldots(i)$$

and

$$3x-2y+17=0 \qquad \ldots(ii)$$

Solving (*i*) and (*ii*) simultaneously, we have

$$\frac{x}{51-64}=\frac{y}{-96-34}=\frac{1}{-4-9}$$

or

$$\frac{x}{-13}=\frac{y}{-130}=\frac{1}{-13}$$

$\therefore$ $x=1$ and $y=10$.

Hence the foot of the perpendicular is (1, 10).

**Example 3.** *Find the distance between the parallel lines*

$$3x+4y-5=0 \qquad \ldots(1)$$

and

$$6x+8y-45=0 \qquad \ldots(2)$$

*(1947)*

The distance between two parallel lines is the distance of any point on one of the lines from the other.

Putting $y=0$ in (1), we get $x=\frac{5}{3}$.

$\therefore$ $(\frac{5}{3}, 0)$ is a point on (1).

Its perpendicular distance from (2)

$$=\frac{\frac{5}{3}\times 6+0-45}{\sqrt{6^2+8^2}}=\frac{-35}{10}=\tfrac{7}{2} \text{ (numerically)}$$

which is the distance between the two lines.

## Exercise IV (c)

1. Find the perpendicular distance of the point

   (*i*) $(2, -1)$ from the line $3x-4y=5$

   (*ii*) $(1, 1)$ from the line $3x+4y+8=0$

   (*iii*) $(b, a)$ from the line $a(x-a)=b(y-b)$

   (*iv*) $(b, a)$ from the line $\dfrac{x}{a}-\dfrac{y}{a}=1$

   (*v*) $(0, 0)$ from the line $h(x+h)+k(y+k)=0$.

2. If $p$ is the perpendicular distance of the origin from a line whose intercepts on the axes are $a$ and $b$ show that

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

3. (*i*) Find the length of the perpendicular from the point $(2, 3)$ on the line $3x-4y+13=0$.

   (*ii*) Also find the co-ordinates of the foot of the perpendicular.

   (*iii*) And the equation of this perpendicular.

4. Find the feet of the perpendiculars from $(1, 1)$ to the lines $x-2y+2=0$ and $2x-y+1=0$. Also find the length of the perpendicular from $(1, 1)$ on the line joining these feet.

5. Find the lengths of the altitudes of a triangle whose altitudes are $(0, 0)$, $(1, -1)$ and $(3, 2)$. (*P.U. 1938*)

6. Find the distance between the parallel lines

   (*i*) $3x+4y+15=0$ and $3x+4y-9=0$

   (*ii*) $3x+4y-5=0$ and $6x+8y-45=0$ (*P.U. 1947*)

   (*iii*) $ax+by+c=0$ and $ax+by+c'=0$

   (*iv*) $y=mx+c$ and $y=mx+d$. (*P.U. 1944*)

7. Find the points on the line $y=x$ which are at a distance of 5 units from the line

$$4x+3y-1=0.$$

8. Find the equations of straight lines parallel to $3x+4y=7$ and at a distance 4 from the point (2, 1).

9. Find the point on the line $4x+3y=1$ which is nearest to (2, 6).

10. The equations of the sides of a triangle are $3x+y-4=0$, $3x-5y+34=0$ and $3x-2y+1=0$. Find the lengths of the altitudes.

### 4.4. Position of points with respect to a line.

*Two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie on the same or opposite sides of the line $ax+by+c=0$ according as the expressions $ax_1+by_1+c$ and $ax_2+by_2+c$ have the same or opposite signs.*

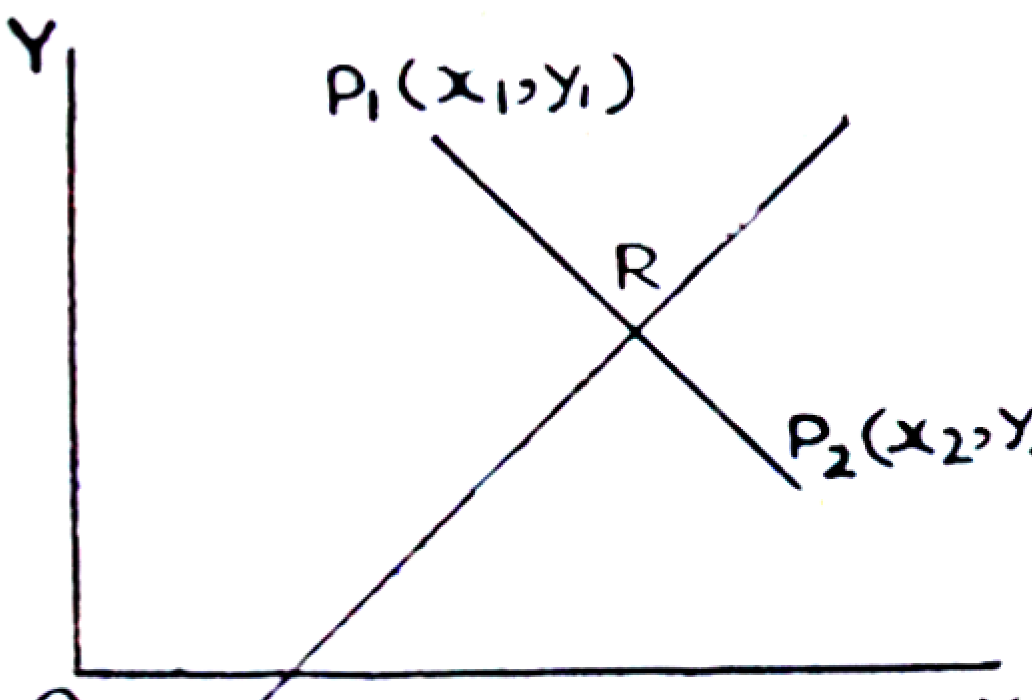



FIG 1

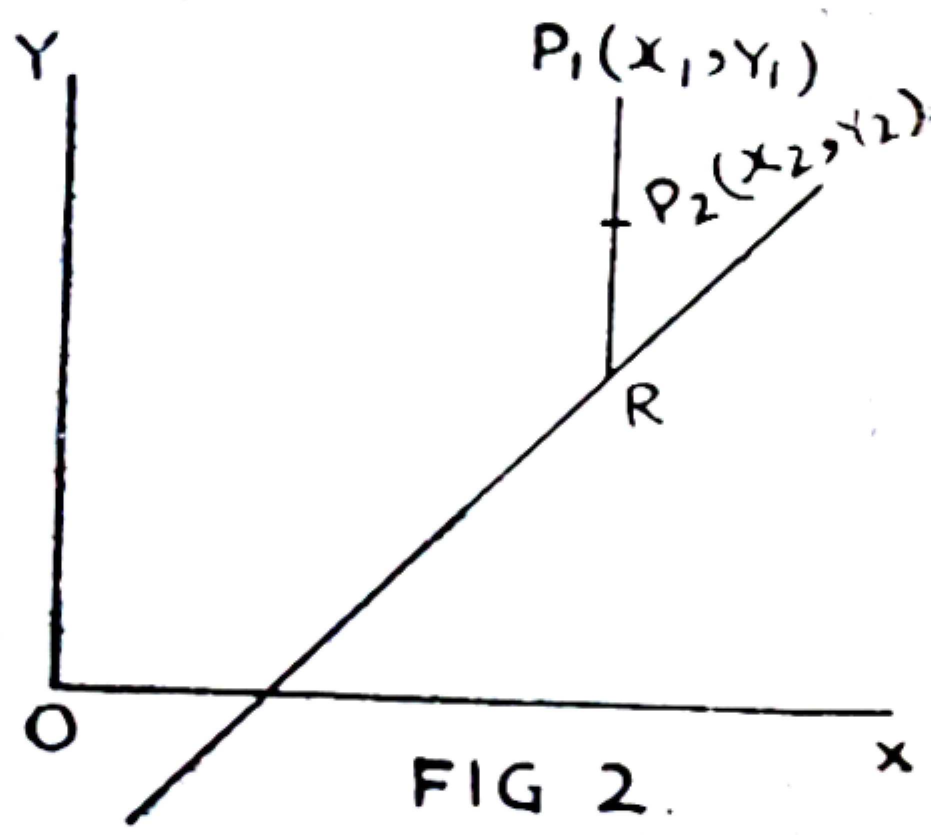



FIG 2.

Let the line $P_1P_2$ meet the given line in R. Let R divide $P_1P_2$ in the ratio $m_1 : m_2$.

Then the co-ordinates of R are $\left(\dfrac{m_1x_2+m_2x_1}{m_1+m_2}, \dfrac{m_1y_2+m_2y_1}{m_1+m_2}\right)$

$\because$ R lies on the given line

$$\therefore \quad a \cdot \frac{m_1x_2+m_2x_1}{m_1+m_2}+b \cdot \frac{m_1y_2+m_2y_1}{m_1+m_2}+c=0$$

*i.e.*, $m_1(ax_2+by_2+c)+m_2(ax_1+by_1+c)=0$

or $\dfrac{m_1}{m_2}=-\dfrac{ax_1+by_1+c}{ax_2+by_2+c}.$

If $P_1$ and $P_2$ lie on the opposite sides of the given line (Fig. 1), then $\dfrac{m_1}{m_2}$ is positive as R lies between $P_1$ and $P_2$ in that case.

Hence the expressions $ax_1+by_1+c$ and $ax_2+by_2+c$ must have opposite signs.

But if $P_1$ and $P_2$ lie on the same side of the given line (Fig. 2), then $\dfrac{m_1}{m_2}$ is negative, as in this case R does not lie between $P_1$ and $P_2$.

Hence the expressions $ax_1+by_1+c$ and $ax_2+by_2+c$ must have the same sign.

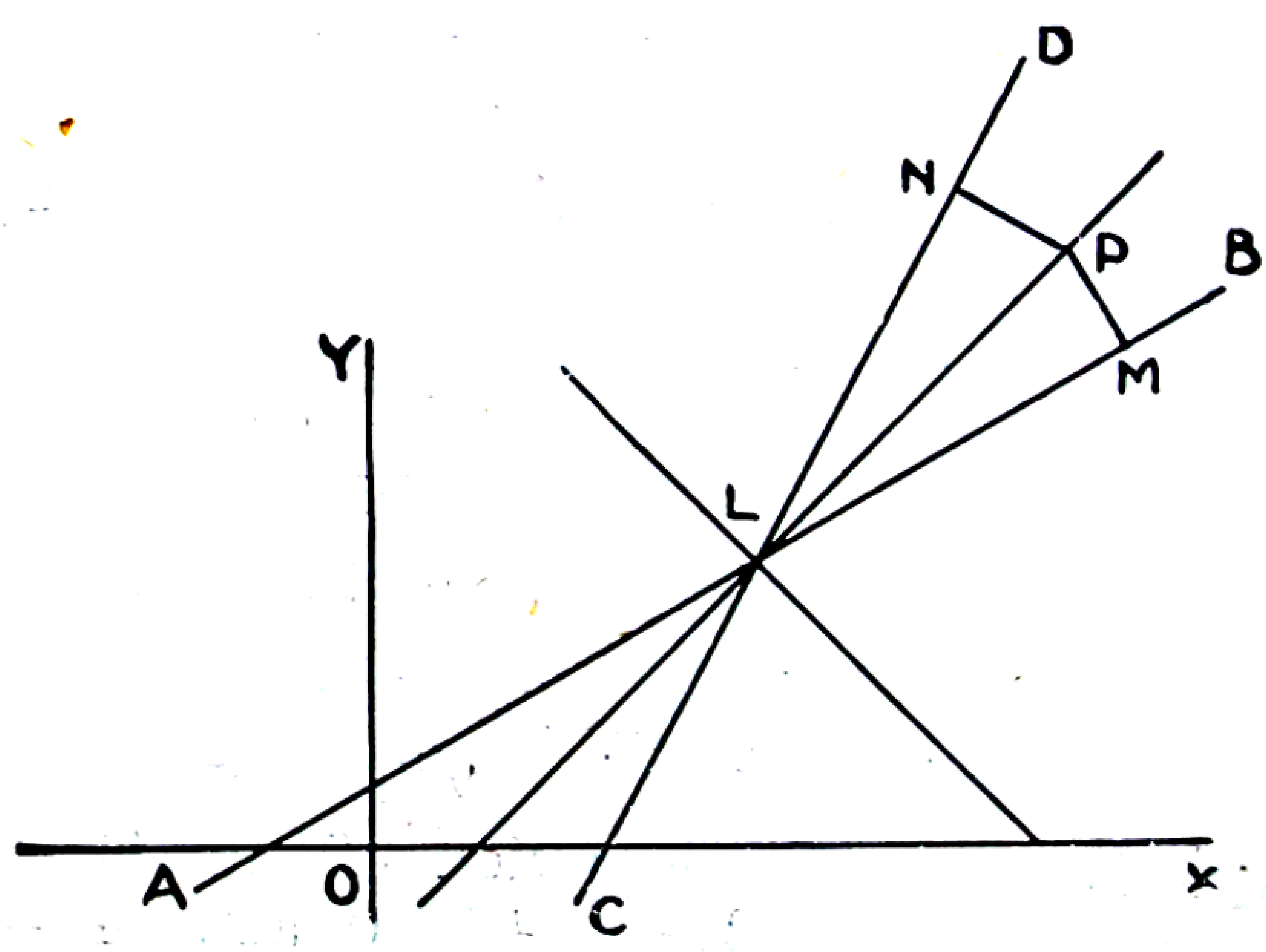
Hence the points $(x_1, y_1)$ and $(x_2, y_2)$ lie on the same or opposite sides of the line $ax+by+c=0$ according as $ax_1+by_1+c$ and $ax_2+by_2+c$ have the same or opposite signs.

### 4.5. Bisectors of angles between two lines.

Let the equations of the lines be

$$a_1x+b_1y+c_1=0 \qquad \ldots(1)$$
$$a_2x+b_2y+c_2=0 \qquad \ldots(2)$$

The equations being written such that $c_1$ and $c_2$ are both positive.




Let P$(x, y)$ be any point on either bisector.

Then the length of the perpendicular from P on the two given lines is the same.

$$\therefore \quad \frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}} \qquad \ldots(3)$$

$\therefore$ The two bisectors are

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=+\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}} \qquad \ldots(4)$$

and $$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=-\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}} \qquad \ldots(5)$$

**Note.** Sometimes we have to distinguish between the bisector of the acute angle and that of the obtuse angle. This is done by taking one of the bisectors (4) and (5), and finding the angle that tnis bisector makes with either of the given lines. If the angle is less than 45°, then the bisector chosen is of the acute angle. Otherwise it is of the obtuse angle.

**4.51. The bisector of the angle in which origin lies.**

Let the origin fall with n the angle ALC. Then the perpendiculars to the lines from any point within the angle ALC will have the same sign as the perpendiculars from the origin.

Now since $c_1$, $c_2$ are both positive. $a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$ are both positive for the points within the angle ALC,

$\therefore$ The equation of the bisector of the angle ALC, in which they lie, is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

and the equation of the bisector of the other angle is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a^2+b^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

**Example 1.** Find the equation of the bisector of that angle between the lines $3x-4y+7=0$ and $4x-3y-17=0$, which contains the origin.

The equations can be written as follows after making the constant terms positive

$$3x-4y+\ 7=0 \qquad \text{..(1)}$$
$$-4x+3y+17=0 \qquad \text{...(2)}$$

Then the equation of the required bisector is

$$\frac{3x-4y+7}{5}=\frac{-4x+3y+17}{5}$$

or $$7x-7y-10=0$$

**Example 2.** Find the equation of the bisector of the acute angle between the lines $2x-y-4=0$ and $x-2y+10=0$.

The equations of the two given lines are

$$2x-y-\ 4=0 \qquad \text{...(1)}$$
$$x-2y+10=0 \qquad \text{...(2)}$$

The equations of the two bisectors are

$$\frac{2x-y-4}{\sqrt{5}}=\pm\frac{x-2y+10}{\sqrt{5}}$$

*i.e.*, $$x+y-14=0 \qquad \text{...(3)}$$
and $$3x-3y+6=0 \qquad \text{...(4)}$$

Out of these two bisectors, we have to find which bisects the acute angle between (1) and (2).

Let $\theta$ be the angle between (1) and (3).

Then $\tan\theta=\dfrac{2.1-1(-1)}{2+1.(-1)}=3$

$\therefore$ $\theta$ is greater than 45°. Hence (2) bisects the obtuse angle between (1) and (2). There (4) is the bisector of the acute angle between (1) and (2), *i.e.*, the required bisector is

$$x-y+2=0.$$

## Exercise IV (d)

1. Examine whether the points (4, 2) and (3, —4) lie on the same side of the line $3x+2y=5$ as the origin.

2. Show that the origin lies inside the triangle whose vertices are (1, 1) (—3, —1) and (2, —5).

3 Show that the points (1, 1) and (2, —1) lie on the same side of the line $2x-5y+7=0$, whereas points (2, —1) and (3, 4) lie on the opposite sides.

4. Find the locus of a point which moves so that the perpendiculars drawn from it to the lines

(i) $3x+4y=5$ and $12x-5y=13$. (P. U.)

(ii) $x \cos \alpha+y \sin \alpha=p$ and $x \cos \beta+y \sin \beta=p'$ are equal. (P. U. 1942)

5. Find the bisectors of the angle which contain the origin between the following pairs of the lines :—

(i) $3x+4y-27=0$ and $5x-12y-16=0$

(ii) $2x-y+2=0$ and $x+2y+7=0$

(iii) $5x+y-7=0$ and $y-5y+7=0$

6. Find the bisector of the acute angle between the following pairs of lines :—

(i) $3x-4y+12=0$ and $5x+12y-60=0$

(ii) $3x-4y+13=0$ and $12x+5y-32=0$

(iii) $12x+5y=4$ and $24y-7x=9$

7. Find the incentre of the triangle formed by the lines $y+1=0$, $4x-3y-7=0$ and $8x-15y+49=0$.

## Revision Exercise I

1. A line is of length 10 and one end is at the point (2, $-3$). If the abscissa of the other end be 10, prove that its ordinate must be 3 or $-9$.

2. Prove that the points (2, $-2$), (8, 4), (5, 7) and ($-1$, 1) are the angular points of a rectangle.

3. Prove that the point $\left(\frac{-1}{14}, \frac{19}{14}\right)$ is the centre of the circle circumscribing the triangle whose angular points are (1, 1), (2, 3) and ($-2$, 2).

4. The line joining the points ($-6$, 8) and (8, $-6$) is divided into four equal parts. Find the co-ordinates of the points of section.

5. Prove that the co-ordinates $x$ and $y$ of the middle point of the line joining the point (2, 3) to the point (3, 4) satisfy the equation $x-y+1=0$.

6. Find the centroid of the triangle whose angular points are $(3, -5)$, $(-7, 4)$ and $(10, -2)$ respectively.

7. A, B, C are the points $(-1, 5)$, $(3, 1)$ and $(5, 7)$ respectively. D, E, F are the middle points of BC, CA, AB respectively. Prove that

$$\triangle ABC = 4 \triangle DEF.$$

8. A point moves so that its distance from the axis of $x$ is half its distance from the origin. Find the equation of its locus.

9. A point moves so that twice its distance from the axis of $y$ exceeds its distance from the axis of $x$ by 2. Find the equation of its locus.

10. Find the points which are at a distance 5 from $(3, 4)$ and at a distance 13 from $(5, 12)$.

11. Prove that the straight line which makes an angle $\tan^{-1} 5$ with the axis of $x$, and which cuts the axis of $y$ in the point $(0, -5)$ passes through the point $(1, 0)$.

12. The equation $x+y+5=0$ is equivalent to

$$x \cos \frac{5\pi}{4} + y \sin \frac{5\pi}{4} = \frac{5}{\sqrt{2}}$$

13. When the equation $5x+12y-26=0$ is written in the form

$$x \cos \alpha + y \sin \alpha - p = 0$$ find the value of $p$.

14. Prove that the line through the points $(5, 0)$ $(0, -2)$ passes also through the points

$$(15, 4) \text{ and } (-5, -4)$$

15. Prove that the line $y-x+2=0$ cuts the line joining $(3, -1)$ and $(8, 9)$ in the ratio $2 : 3$.

16. Prove that the line through the two points $(9, 3)$ and $(15, -3)$ cuts off equal intercepts from the axes.

17. Show that the four points $(0, 0)$ $(-1, 1)$, $(-7, -4)$ and $(9, 6)$ are in the four different compartments made by the two straight lines

$$2x-3y+1=0 \text{ and } 3x-5y+2=0.$$

18. Find the equations of the sides of the triangle, the co-ordinates of whose vertices are (1, 2), (2, 3) and (–3, –5).

19. Find the acute angle between the lines
$$3x+y-7=0 \text{ and } x+2y+9=0.$$

20. Find the lines through (2, 3) which make acute angles of 45° with the line
$$3x-5y+5=0.$$

21. Find the perpendicular distances of the point (2, 3) from the lines
$$4x+3y-7=0,\ 5x+12y-20=0 \text{ and } 3x+4y-8=0.$$

22. Find the equations of two straight lines which are parallel to $x+7y+2=0$ and at unit distance from the point (1, –1).

23. Find the equation of the line joining (1, 1) to the point of intersection of the lines
$$3x+4y-2=0 \text{ and } x-2y+5=0.$$

24. Find the area of the triangle formed by the lines
$$y-x=0,\ y+x=0,\ x-c=0.$$

25. Show that the area of the triangle formed by the lines $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$ is
$$\tfrac{1}{2}\,\frac{(c_1-c_2)^2}{m_2-m_1}.$$

26. A triangle is formed by the lines $3x+4y-6=0$, $12x-5y-3=0$ and $4x-3y+12=0$. Find the internal bisector of the vertical angle opposite to the side $3x+4y-6=0$.

(*P.U. 1933*)

27. OA and OB are two fixed perpendicular st. lines. The line AB moves such that OA+OB=8. Find the locus of the middle point of AB.

28. Find the locus of the foot of the perpendicular from the origin on a line which always passes through a point $(h, k)$.

(*P.U. 1935*)

29. Find the incentre of the triangle whose sides are
$$3x+4y-12=0,\ 3x-4y-36=0 \text{ and } x=0.$$

30. Find the orthocentre of the triangle whose sides are
$$y=2x,\ 2y=x \text{ and } x+y+9=0.$$

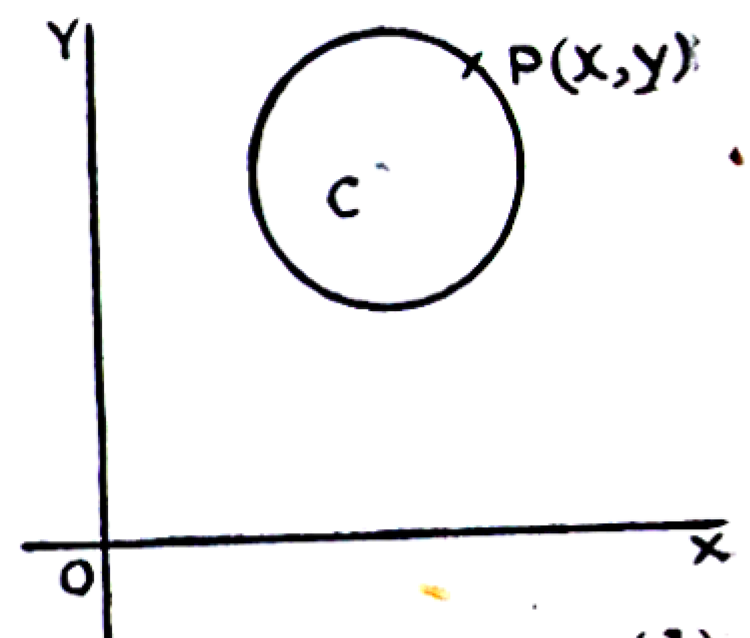
CHAPTER V

# THE CIRCLE

**5.1. Definition.** *The circle is the locus of a point which moves in a plane such that its distance from a fixed point, remains constant. The fixed point is called the* **centre**, *and the constant distance, the* **radius** *of the circle.*

Hence a circle is specified when the position of its centre and its radius are known.

## 5.11. The equation of a circle, its centre and radius being given.




Let C $(h, k)$ be the centre and $a$ the radius of the circle.

Let P $(x, y)$ be any point on the circle.

Then $PC=a$ or $PC^2=a^2$

But $PC^2=(x-h)^2+(y-k)^2$

Hence the equation of the circle is

$$(x-h)^2+(y-k)^2=a^2 \qquad \ldots(1)$$

**5.12.** If the centre of the circle is at the origin, $h=k=0$, and, therefore the equation of the circle is

$$x^2+y^2=a^2 \qquad \ldots(2)$$

This is the simplest form of the equation.

**5.13.** *To prove that the equation*

$$x^2+y^2+2gx+2fy+c=0$$

*represents a circle and to find its centre and radius.*

The given equation may be written as

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$$

or $$(x+g)^2+(y+f)^2=(\sqrt{g^2+f^2-c})^2$$

This shows that the distance of any point $(x, y)$ on the locus from the fixed point $(-g, -f)$ is constant and equal to $\sqrt{g^2+f^2-c}$.

Hence the locus represented by the given equation is a circle whose centre is $(-g, -f)$ and the radius is $\sqrt{g^2+f^2-c}$.

There are three possibilities :—

(i) If $g^2+f^2-c$ is positive the locus is a real circle of centre $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.

(ii) If $g^2+f^2-c=0$, the radius vanishes and locus is a point circle.

(iii) If $g^2+f^2-c$ is negative, the locus is an imaginary circle.

## 5.14. Characteristics of the equation of a circle.

In order that an equation may represent a circle it must be possible to write it in form (1) of article 5.11.

*Hence if an equation is to represent a circle :*

(i) *it must be of the second degree,*

(ii) *the co-efficients of $x^2$ and $y^2$ should be equal,*

(iii) *there should be no term involving the product $xy$.*

**Note.**—The equation $x^2+y^2+2gx+2fy+c=0$ is taken as the general equation of a circle. In this equation if $c=0$, the circle passes through the origin.

**Example 1.** Find the centre and radius of the circle

$$2x^2+2y^2+8x-12y+18=0$$

Divide the equation by 2 to make the co-efficient if $x^2$ or $y^2$ unity

*i.e.*
$$x^2+y^2+4x+6y+9=0$$

The co-ordinates of the centre now are obtained by dividing the co-efficients of $x$ and $y$ by $-2$ respectively

*i.e.* They are $(-2, -3)$

and radius$=\sqrt{g^2+f^2-c}=\sqrt{4+9-9}=2.$

**Example 2.** Find the equation of a circle having radius 3 and concentric with the circle $x^2+y^2-8x+6y+9=0$

The centre of the given circle is $(4, -3)$.

This is also the centre of the required circle.

Hence its equation is

$$(x-4)^2+(y+3)^2=9$$

or $$x^2+y^2-8x+6y+16=0$$

**Second Method.** Equation of any circle concentric with the given circle is

$$x^2+y^2-8x+6y+k=0 \qquad \ldots(1)$$

Radius of circle $(1)=\sqrt{16+9-k}$

But its radius $=3$

$\therefore \qquad \sqrt{16+9-k}=3$

or $\qquad 25-k=9 \qquad$ or $\quad k=16$

$\therefore$ the equation of the required circle is

$$x^2+y^2-8x+6y+16=0.$$

**5.15.** The general equation of a circle

$$x^2+y^2+2gx+2fy+c=0$$

contains three arbitrary constants $g, f, c$. Hence to find the equation of a circle, we must know the values of these constants. But to determine the values of three constants, we require three independent equations involving them, each equation being an algebraical expression of some geometrical condition about the particular circle. Hence a circle is fixed if three independent conditions about it are known.

**Example 1.** *Find the equation of the circle which passes through the points : $(2, -1)$ ; $(2, 3)$ ; $(4, -1)$.*

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(1)$$

Points $(2, -1)$, $(2, 3)$ and $(4, -1)$ lie on the circle.

$$\therefore \quad 5+4g-2f+c=0 \qquad \ldots(2)$$

$$13+4g+6f+c=0 \qquad \ldots(3)$$

$$17+8g-2f+c=0 \qquad \ldots(4)$$

To find the values of $g$, $f$, $c$, we have to solve these equations simultaneously.

Subtracting (2) from (4)

$$12+4g=0 \qquad \text{or} \quad g=-3 \qquad \ldots(5)$$

Subtracting (2) from (3)

$$8+8f=0 \quad \text{or} \quad f=-1 \qquad \ldots(6)$$

Substituting the values of $f$ and $g$ in (2)

$$5-12+2+c=0$$

or $$c=5 \qquad \ldots(7)$$

Hence the required equation is

$$x^2+y^2-6x-2y+5=0.$$

**Example 2.** *Find the equation of the circle which passes through (1, 1) and (2, 2) and has a radius of one unit.*

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(1)$$

Points (1, 1) and (2, 2) lie on the circle

$\therefore$ $$2+2g+2f+c=0 \qquad \ldots(2)$$

and $$8+4g+4f+c=0 \qquad \ldots(3)$$

Also the radius is 1,

$\therefore$ $$g^2+f^2-c=1 \qquad \ldots(4)$$

To solve the equations 2, 3, 4 simultaneously, multiply (2) by 2 and subtract from (3).

We get $4-c=0$ or $c=4$ ...(5)

Substituting this value in (2), we get

$$2+2g+2f+4=0$$

*i.e.,* $$f=-(g+3) \qquad \ldots(6)$$

Substituting in (4), the values obtained in (5) and (6)

$$g^2+(g+3)^2-4=1 \quad \text{or} \quad 2g^2+6g+4=0$$

$$\text{or} \quad g^2+3g+2=0$$

or $g=-2$ and $g=-1$

Hence from (6) $f=-2$ when $g=-1$

and $f=-1$ when $g=-2$

There are two sets of values of $f$ and $g$. Hence there are two circles satisfying the given conditions.

The required equations are

$$x^2+y^2-2x-4y+4=0$$

and $$x^2+y^2-4x-2y+4=0.$$

**Example 2.** *Find the equation of the circle which passes through the points (4, 1) and (6, 5) and has its center on the line.*

$$4x+y-16=0.$$

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots(1)$$

Points (4, 1) and (6, 5) lie on the circle.

$\therefore \qquad 17+8g+2f+c=0 \qquad \ldots(2)$

and $\qquad 61+12g+10f+c=0 \qquad \ldots(3)$

Also, the centre $(-g, -f)$ lies on the line $4x+y-16=0$

$\therefore \qquad -4g-f-16=0 \qquad \ldots(4)$

Equations 2, 3, 4 are to be solved simultaneously for $f$, $g$ and $c$.

Subtracting (2) from (3), we get

$$44+4g+8f=0 \qquad \ldots(5)$$

Adding (4) and (5), $\quad 28+7f=0 \quad$ or $\quad f=-4$.

$\therefore$ From (4), $\quad g=-3$ and then from (2), $c=15$

Hence the required equation is

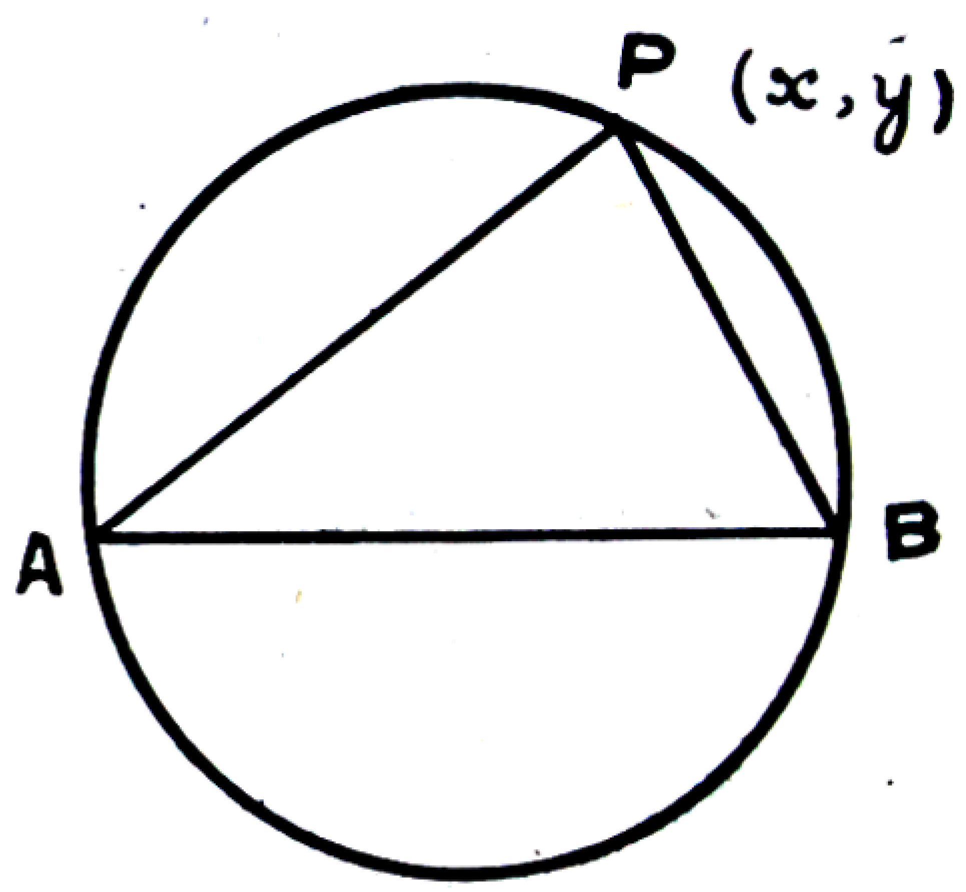
$$x^2+y^2-6x-8y+15=0$$

**5.16. The equation of the circle on the join of A($x_1$, $y_1$) and B($x_2$, $y_2$) as diameter.**

Let P($x$, $y$) be any point on the circle.

$\angle$APB is a right angle being an angle in a semi-circle.

$\therefore$ PA and PB are perpendicular to each other. Hence the product of their slopes is $-1$.




The slope of PA$=\dfrac{y-y_1}{x-x_1}$

The slope of PB$=\dfrac{y-y_2}{x-x_2}$

$$\therefore \qquad \frac{y-y_1}{x-x_1}\times\frac{y-y_2}{x-x_2}=-1$$

$$\therefore \quad (y-y_1)(y-y_2)=-(x-x_1)(x-x_2)$$

*i.e.* $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
which is the required equation.

## Exercise V (a)

1. Find the equation to a circle whose
   (*i*) centre is $(3, -2)$ and radius 4.
   (*ii*) centre is $(a, a)$ and diameter $a$.
   (*iii*) centre is $(3, -5)$ and which passes through $(1, 2)$.

2. Find the centre and radius of the circle
   (*i*) $x^2+y^2-49=0$
   (*ii*) $x^2+y^2+8x-6y-24=0$
   (*iii*) $2x^2+2y^2-12x+16y+18=0$
   (*iv*) $x^2+y^2-ax-by=0$
   (*v*) $x^2+y^2+2x+1=0$
   (*vi*) $3x^2+3y^2-5x-6y+4=0$
   (*vii*) $5x^2+5y^2+4x-8y-16=0$
   (*viii*) $4x^2+4y^2=12ax+6ay+a^2$.

3. Find the equation of a circle whose centre is 2, 3 and which passes through the centre of the circle $x^2+y^2-4x+7=0$.

4. Find the equation of the circle whose centre is $(4, 5)$ and whose circumference passes through the centre of the circle $x^2+y^2+4x-6y-12=0$.

5. Find the equation of the circle which passes through the centre of the circle $x^2+y^2+8x+10y-33=0$ and is concentric with $x^2+y^2-4x-6y+11=0$.

6. Find the equation of the circle whose
   (*i*) centre is $(2, 3)$ and which touches the line
   $$3x+4y+2=0.$$
   (*ii*) centre is $(4, 5)$ and which touches the line
   $$5x+12y+11=0.$$

7. One end of the diameter of the circle $x^2+y^2-8x-10y+1=0$ is $(-2, 3)$. Find the other end.

8. Find the equation of the circle which passes through the points :

(i) $(1, 2)$ ; $(2, 1)$ ; $(0, 0)$

(ii) $(0, 0)$ ; $(0, 3)$ ; $(-4, 0)$ (*P.U.*)

(iii) $(2, 3)$ ; $(3, 2)$ ; $(5, 1)$

(iv) $(1, 2)$ ; $(2, 1)$ ; $(2, 3)$ (*P.U. 1936*)

(v) $(-3, 4)$ ; $(-2, 0)$ ; $(1, 5)$. (*P.U. 1937*)

(vi) $(1, 1)$ ; $(2, -1)$ ; $(3, -2)$. (*P.U. 1941*)

9. Find the equation of the circle which passes through the points $(0, 0)$, $(a, 0)$ and $(0, b)$ ; find its centre and the radius. (*P.U. 1945 S*)

10. Find the equation of the circle circumscribing the triangle whose sides are

$$x+2y=0, \quad x-3y+1=0, \quad 3x+y-5=0.$$

11. Find the equation of the circle whose diameter is the line joining the points :

(i) $(3, 4)$ and $(2, -7)$.

(ii) $(4, -3)$ and $(6, 3)$.

(iii) $(a \cos \theta, b \sin \theta)$ and $(-a \sin \theta, b \cos \theta)$.

12. Find the equation of the circle which

(i) passes through the points $(2, 3)$, $(4, -1)$ and has its centre on the line $y=1$.

(ii) passes through the points $(-1, 2)$, $(3, -2)$ and has its centre on the line $x=2y$.

(iii) passes through the points $(a, 0)$, $(0, b)$ and has its centre on the axis of $x$.

13. Find the equation of the circle which passes through the points $(2, 3)$ and $(6, -1)$ and whose radius is equal to 4.

14. Find the equation of the circle which

(i) touches each axis at a distance 5 from the origin,

(ii) touches each axis and is of radius $a$,

(iii) touches both axes of ~~x~~ and passes through the point $(-2, -3)$,

(*iv*) touches the axis of $x$ and passes through the points $(1, -2)$ and $(3, -4)$.

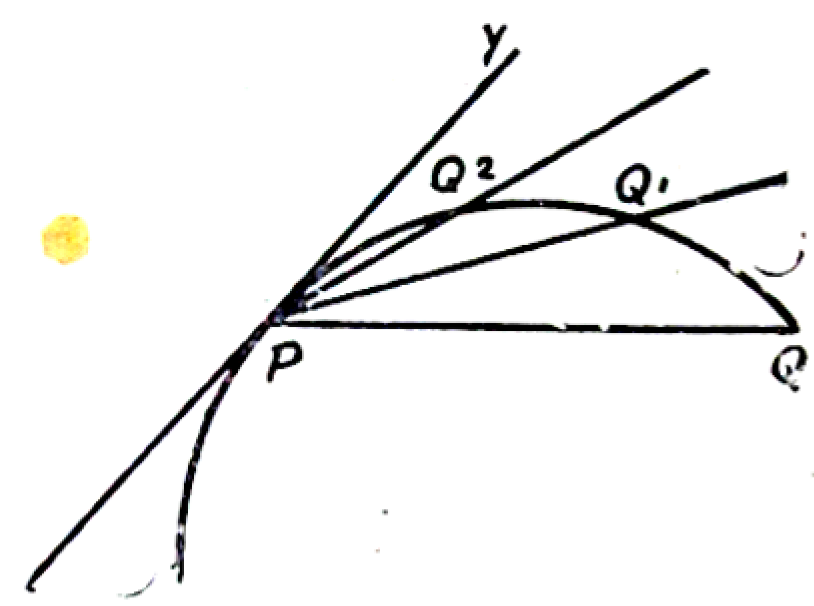
15. Find the equation of the circle which passes through $(1, 0)$ and $(2, 0)$ and touches the line $y=x$.

16. Find the equation of the circle which passes through the origin and the point $(2, 1)$ and touches $y=x$ at the origin. (*P.U. 1944*)

17. Find the equation of the circle which passes through $(4, 0)$, $(9, 0)$ and touches $y=2x$. (*P.U. 1943*)

## Tangents and Normals

**5.2. Tangent. Def.** Let P and Q be any two points near one another, on any curve. Join PQ; then PQ is called a **secant.**




Let the secant PQ be rotated about P, so that the second point of intersection Q gradually moves up to P along the curve. The limiting position of the line PQ when Q finally tends to coincide with P is called the **tangent** to the curve at P.

Thus a tangent is the limiting position of a secant when its two points of intersection coincide.

The point P is called the **point of contact** of the tangent.

**Normal.** *The straight line through the point P (in the point of contact) perpendicular to the tangent at P is called the* **normal** *to the curve at P.*

**5.21. Equation of the chord of a circle joining two given points on the circle $x^2+y^2=a^2$.**

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points on the circle. Then as the points lie on the circle $x^2+y^2=a^2$

$$x_1^2+y_1^2=a^2 \qquad \ldots(1)$$
$$x_2^2+y_2^2=a^2 \qquad \ldots(2)$$

Subtracting (2) from (1)

$$(x_1^2-x_2^2)+(y_1^2-y_2^2)=0$$

or $(x_1-x_2)(x_1+x_2)=-(y_1-y_2)(y_1+y_2)$

or $$\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2} \qquad ...(3)$$

Hence the slope of

$$PQ=\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}$$

$\therefore$ the equation of PQ is

$$y-y_1=-\frac{x_1+x_2}{y_1+y_2}(x-x_1) \qquad ...(4)$$

**5.22. Equation of the tangent to a circle at a given point.**

Tangent has been defined as the limiting position of the chord PQ as Q tends to coincide with P *i.e.*, as $x_2 \to x_1$ and $y_2 \to y_1$.

Then equation (4) of the preceding article becomes

$$y-y_1=-\frac{2x_1}{2y_1}(x-x_1)$$

or $$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

or $yy_1-y_1^2=-xx_1+x_1^2$

or $yy_1+xx_1=x_1^2+y_1^2=a^2$ from (1)

which is the required equation.

**5.23. Equation of the normal at a given point on a circle.**

The tangent at $P(x_1, y_1)$ to the circle $x^2+y^2=a^2$ is

$$xx_1+yy_1=a^2$$

$\therefore$ Slope of the tangent $=\dfrac{-x_1}{y_1}$

$\therefore$ Slope of the normal which is perpendicular to the tangent $=\dfrac{y_1}{x_1}$

$\therefore$ the equation of the normal is

$$y-y_1=\frac{y_1}{x_1}(x-x_1)$$

or $\quad yx_1-x_1y_1=xy_1-x_1y_1$

or $\quad xy_1-yx_1=0.$

**Note.**—It may be observed that the normal passes through (0, 0) *i.e.*, the centre of the circle.

**5.24.** In case the equation of the circle is given in the general form $x^2+y^2+2gx+2fy+c=0$, the equations of the tangent and the normal at any point P can be obtained by following precisely the same procedure.

Take $Q(x_2, y_2)$ any other point on the circle.

$\because$ P and Q lie on the circle,

$$\therefore \quad x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \ldots(1)$$

and
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \ldots(2)$$

Subtracting (2) from (1)

$$(x_1^2-x_2^2)+(y_1^2-y_2^2)+2g(x_1-x_2)+2f(y_1-y_2)=0$$

or
$$(x_1-x_2)(x_1+x_2+2g)+(y_1-y_2)(y_1+y_2+2f)=0$$

$$\therefore \quad \frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f} \qquad \ldots(3)$$

Now slope of $PQ=\dfrac{y_1-y_2}{x_1-x_2}=-\dfrac{x_1+x_2+2g}{y_1+y_2+2f}$

$\therefore$ the equation of PQ is

$$y-y_1=-\frac{x_1+x_2+2g}{y_1+y_2+2f}(x-x_1) \qquad \ldots(4)$$

The chord PQ becomes tangent to the circle at P ; if Q moving along the curve tends to coincide with P *i.e.*, if $x_2 \to x_1$ and $y_2 \to y_1$, then equation (4) becomes

$$y-y_1=-\frac{2x_1+2g}{2y_1+2f}(x-x_1)$$

or
$$(y-y_1)(y_1+f)=-(x_1+g)(x-x_1)$$

or
$$yy_1+fy-y_1^2-fy_1=-(xx_1+gx-gx_1-x_1^2)$$

or
$$yy_1+xx_1+fy+gx=x_1^2+y_1^2+fy_1+gx_1$$

or by adding $gx_1+fy_1+c$ to both the sides

$$yy_1+xx_1+fy+fy_1+gx+gx_1+c=x_1^2+y_1^2+2fy_1+gx_1+c$$

or by (1)

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \qquad \ldots(5)$$

which is the equation of the tangent at $(x_1, y_1)$.

Now the slope of the tangent at $(x_1, y_1)$

$$=-\frac{x_1+g}{y_1+f}$$

$\therefore$ the slope of the normal

$$=\frac{y_1+f}{x_1+g}$$

$\therefore$ the equation of the normal is

$$y-y_1=+\frac{y_1+f}{x_1+g}(x-x_1)$$

or $$\frac{y-y_1}{y_1+f}=\frac{x-x_1}{x_1+g}.$$

It may be seen that the normal given by the above equation passes through the point $(-g, -f)$ *i.e.*, the centre of the circle.

**Note.** The equation of the tangent to any circle (or any other conic) at a point $(x_1, y_1)$ is written as follows :—

*In the equation of the circle (or the conic) write $xx_1$ and $yy_1$ for $x^2$ and $y^2$ respectively, and $x+x_1$ for $2x$ and $y+y_1$ for $2y$, the constant term being kept unchanged.*

**5.25.** The equation of the tangent at a point $(x_1, y_1)$ on the circle $x^2+y^2+2gx+2fy+c=0$ can be formed by following method also :—

Now

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \ldots(1)$$

is any line through P $(x_1, y_1)$.

The distances from P of the points of intersection of the line with the circle are the roots of the quadratic

$$r^2+2r\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}$$
$$+x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \ldots(2)$$

Since $(x_1, y_1)$ lies on the circle the last term in the quadratic is zero and, therefore, one of its roots is zero.

The other root will also be zero and, therefore, the line (1) will be tangent to the circle if

$$(x_1+g)\cos\theta+(y_1+f)\sin\theta=0 \quad ...(3)$$

This equation gives $\tan\theta$, *i.e.*, the slope of the tangent to the circle at $(x_1, y_1)$.

Eliminating $\theta$ between (1) and (3) we get the equation of the tangent in the form

$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0$$

or $\quad xx_1+yy_1+gx+fy-x_1^2-y_1^2-gx_1-fy_1=0$

or $\quad xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1 \quad ...(4)$

Now adding $gx_1+fy_1+c$ to both sides of equation (4) we have as the equation of the tangent

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \quad ...(5)$$

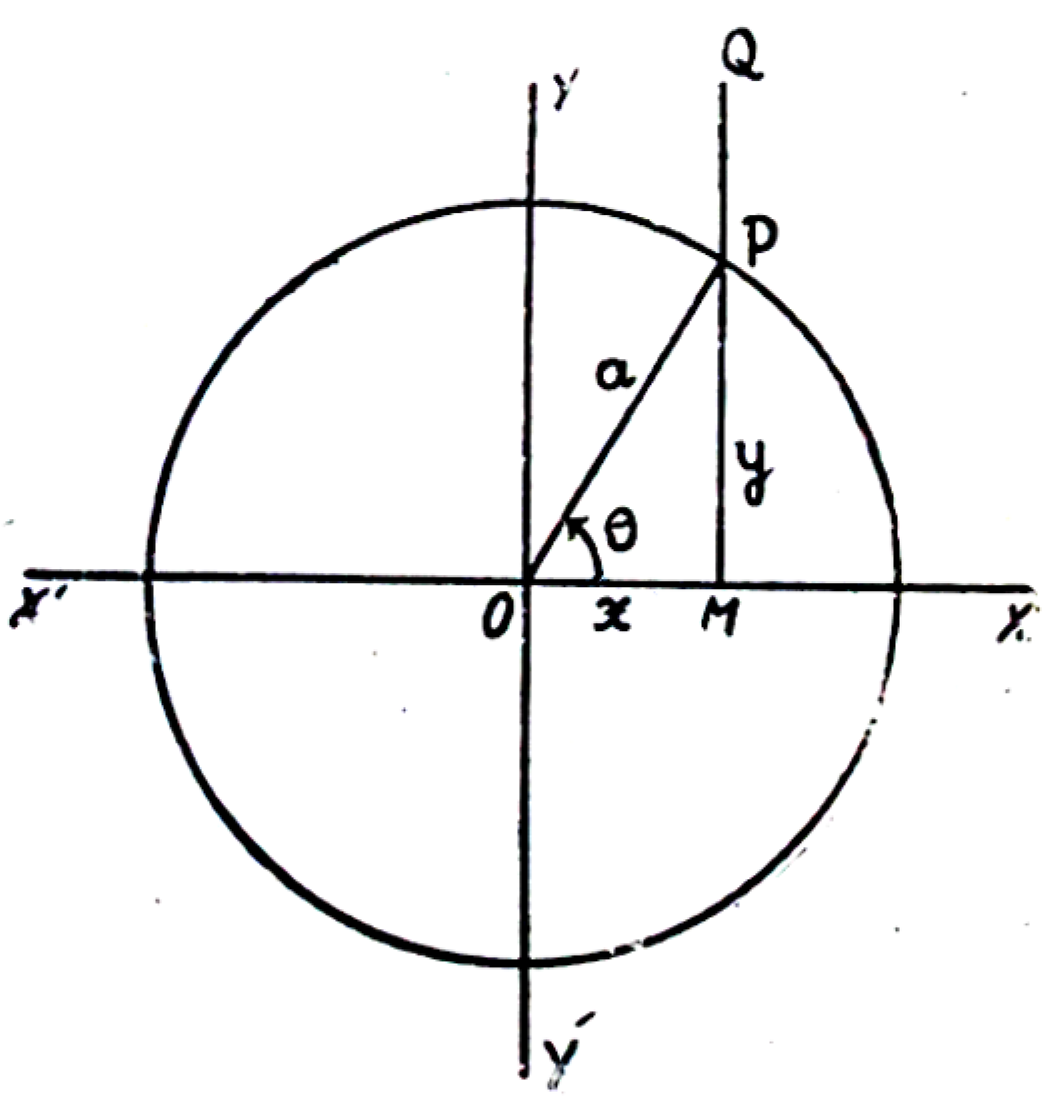
In like manner we can show that equation of tangent at $(x_1, y_1)$ on the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$ ...(6)

## 5·6. Parametric equations of a circle.

It is often possible to express the co-ordinates of a point on a curve in terms of a single variable such that the equation of the curve is identically satisfied by these co-ordinates.

Such co-ordinates are called **parametric** and the single variable is called the **parameter**.

Let the equation of the circle be $x^2+y^2=a^2$. Since $a^2\cos^2\theta+a^2\sin^2\theta=a^2$ for all values of $\theta$, the point $(a\cos\theta, a\sin\theta)$ lies on the circle.




The equations

$$x=a\cos\theta$$
$$y=a\sin\theta$$

are the parametric equations of the circle and $\theta$ is the parameter.

**5·61.** *To find the equation of a chord joining two points $\theta_1$ and $\theta_2$.*

The chord joining the points $(a\cos\theta_1, a\sin\theta_1)$ and $(a\cos\theta_2,$

$a \sin \theta_2)$ is

$$\frac{y-a \sin \theta_1}{a \sin \theta_2-a \sin \theta_1}=\frac{x-a \cos \theta_1}{a \cos \theta_2-a \cos \theta_1}$$

*i.e.,* $$(x-a \cos \theta_1) \cos \frac{\theta_1+\theta_2}{2}+(y-a \sin \theta_1) \sin \frac{\theta_1+\theta_2}{2}=0$$

or $$x \cos \frac{\theta_1+\theta_2}{2}+y \sin \frac{\theta_1+\theta_2}{2}=a \cos \frac{\theta_1-\theta_2}{2}.$$

**5·62.** The equation of the tangent at $\theta_1$ is obtained by putting $\theta_2=\theta_1$.

The tangent at $\theta_1$ is $x \cos \theta_1+y \sin \theta_1=a$.

The equation of the normal at $\theta_1$ is

$$x \sin \theta_1-y \cos \theta_1=0$$

**Example 1.** *Find the equations of the tangent and the normal to the circle $x^2+y^2=25$ at the point (3, 4).*

The equation of the tangent is $3x+4y-25=0$.

The equation of the normal at (3, 4) is

$$4x-3y=4(3)-3.4=0.$$

or $$4x-3y=0.$$

**Example 2.** *Find the equation of the tangent and the normal at the point (2, −2) to the circle*

$$x^2+y^2-6x-3y-2=0$$

The equation of the tangent at (2, −2) is

$$x \times 2+y(-2)-3(x+2) \frac{-3}{2}(y-2)-2=0$$

*i.e.,* $$2x-2y-3x-6-\tfrac{3}{2}y+3-2=0$$

$$-x-\tfrac{7}{2}y-5=0$$

or $$2x+7y+10=0.$$

The equation of the normal at (2, −2) is

$$7(x-2)-2(y+2)=0$$

*i.e.,* $$7x-2y-18=0.$$

**Example 3.** *Find the equations of the tangents to the circle $x^2+y^2-4x-6y-12=0$ which are parallel to the line*

$$3x-4y-7=0.$$

Any st. line parallel to the given line is

$$3x-4y+k=0.$$

This line is a tangent to the circle if the perpendicular distance of the centre from this line is equal to the radius.

Hence $\dfrac{3\times2-4\times3+k}{\pm\sqrt{3^2+\sqrt{4^2}}}=5.$

*i.e.*, $k-6=\pm25 \quad \therefore \; k=31$ or $-19.$

Hence the two tangents are

$$3x-4y+31=0 \text{ and } 3x-4y-19=0$$

## Exercise V (b)

Find the equations of the tangent and normal to the circle

1. $x^2+y^2=13$ at the point (2, 3).

2. $x^2+y^2-6x+4y-12=0$ at the point (6, 2).

3. $x^2+y^2-14x-4y-5=0$ at the point (2, $-1$) whose abscissa is 10.

4. Find the equations of the tangents to the circle $x^2+y^2=9$ which are parallel to the line $5x+12y=6$.

(*P. U. 1938*)

5. Find the equations of the tangents to the circle $x^2+y^2=85$ which are perpendicular to the line $3x-4y+5=0$. (*P. U. 1942*)

6. Show that an infinite number of normals can be drawn to a circle through its centre. (*P. U. 1935*)

7. Find the equation of the tangent to the circle $x^2+y^2-3x+10y-15=0$ at the points where $x=4$.

8. Find whether the straight line $x+y=2\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$.

9. Find the value of $p$ so that the straight line

$$x\cos\alpha+y\sin\alpha-p=0 \text{ may touch the circle}$$
$$x^2+y^2-2ax\cos\alpha-2b\,y\sin\alpha-a^2\sin^2\alpha=0.$$

10. Find the equations of the tangents to the circle $x^2+y^2-2x-4y-4=0$ which are (*i*) parallel (*ii*) perpendicular to the line $3x-4y-11=0$.

## Intersection of a line and a circle.

**5·3.** Let the equation of the circle and the line be respectively

$$x^2+y^2=a^2 \qquad \dots (1)$$
$$y=mx+c \qquad \dots (2)$$

A point of intersection of a straight line and a circle is common to both and hence its co-ordinates must satisfy both the equations. Therefore the co-ordinates are obtained by solving the two equations simultaneously.

Substituting the value of $y$ from (1) in (2), we get

$$x^2+(mx+c)^2=a^2$$

or $$x^2(1+m^2)+2cmx+c^2-a^2=0 \qquad \dots (3)$$

This is a quadratic equation in $x$, giving two values of $x$ which may be real and different, equal, or imaginary.

The two values of $x$ (say $x_1$, and $x_2$) are the abscissae of the points of intersection of (1) and (2). The corresponding values $y_1, y_2$ of $y$ can be obtained from (2) by substituting $x_1$, $x_2$ for $x$.

This shows that a straight line intersects a circle in two points which may be real and different, coincident or imaginary.

**5·31. To find the condition that the line $y=mx+c$ may touch the circle $x^2+y^2=a^2$.**

The line $y=mx+c$ will be a tangent to the circle $x^2+y^2=a^2$ if the two points of intersection are coincident, *i.e.*, if the roots of equation (3) in the preceding article are equal. This will be so if the discriminant of the equation is zero *i.e.*, if

$$4\,m^2c^2-4\,(1+m^2)(c^2-a^2)=0$$

or $$a^2m^2+a^2-c^2=0$$

or $$c=\pm a\sqrt{1+m^2}$$

which is the condition that the given line is a tangent to the given circle.

**5·32.** The above condition that a line may be a tangent to a circle can be found in another way as well. We know that any tangent to a circle is perpendicular to the radius of the circle passing through the point of contact. Hence the perpendicular distance of the line from the centre must be equal to the radius.

The centre of the given circle is $(0, 0)$ and its perpendicular distance from the given line whose equation may be written as $mx-y+c=0$ is

$$\frac{c}{\pm\sqrt{1+m^2}}$$

As the radius is $a$, we have

$$\frac{c}{\pm\sqrt{1+m^2}}=a \qquad \text{or} \qquad c=\pm a\sqrt{1+m^2}$$

**5·33.** Substituting $c=\pm a\sqrt{1+m^2}$ in the equation of the line, we get

$$y=mx\pm a\sqrt{1+m^2},$$

which is the equation of the tangent to the circle $x^2+y^2=a^2$ for all values of $m$, $m$ being the slope of the line.

**This is known as the equation of a tangent in the slope form.**

From the above equation it may be noted that for any value of $m$, there are two lines. This shows that two tangents can be drawn to a circle parallel to each other.

**Example 1.** *Find the points of intersection of the line $y=5x+2$ and the circle $x^2+y^2-13x-4y-9=0$.*

The two equations are $y=5x+2$ ... (1)

and $x^2+y^2-13x-4y-9=0$ ... (2)

Substituting the value of $y$ from (1) in (2),

$$x^2+(5x+2)^2-13x-4(5x+2)-9=0$$

or $26x^2-13x-13=0$

or $2x^2-x-1=0$ *i.e.*, $x=1$, $x=-\frac{1}{2}$

Substituting these values of $x$ in (1)

$$y=7 \text{ when } x=1$$

and $$y=-\tfrac{1}{2} \text{ when } x=-\tfrac{1}{2}.$$

Hence the points of intersection are $(1, 7)$ and $(-\frac{1}{2}, -\frac{1}{2})$.

**Example 2.** *Show that the line $x-y=2$ is a tangent to the circle $x^2+y^2=2$. Find also the co-ordinates of the point of contact.*

Substituting the value of $x$ from the equation of the line in the equation of the circle, we get

$$(y+2)^2+y^2=2$$

or $$2y^2+4y+2=0$$

or $$y^2+2y+1=0 \qquad \textit{i.e.}, y=-1.$$

This gives us only one value of $y$. The corresponding value of $x$, then is 1.

There being only one point of intersection *i.e.*, $(1, -1)$, the line is a tangent. $(1, -1)$ is the point of contact.

*The line could be proved to be a tangent to the circle also by showing that the radius of the circle is equal to the perpendicular distance from the centre to the line.*

**Example 3.** *Find the equations of the tangents to the circle $x^2+y^2=25$ parallel to the line $3x+2y-1=0$.*

Equation of any tangent to the given circle in slope form is

$$y=mx\pm5\sqrt{1+m^2} \qquad \ldots (1)$$

This tangent is parallel to the line $3x+2y-1=0$.

Slope of the given line $=-\frac{3}{2}$

$$\therefore \qquad m=-\tfrac{3}{2}$$

$\therefore$ substituting the value of $m$ in (1), we get

$$y=-\tfrac{3}{2}x\pm5\sqrt{1+\tfrac{9}{4}}$$

or $$2y=-3x\pm5\sqrt{13}$$

$\therefore$ the equations of the required tangents are

$$3x+2y+5\sqrt{13}=0$$

and $$3x+y-5\sqrt{13}=0.$$

**5·33. Intercept on a line cut off by a circle.**

Let the circle be $\quad x^2+y^2=a^2 \quad$ ...(1)

and the line be $\quad y=mx+c \quad$ ...(2)

Substituting the value of $y$ from (2) in (1) and simplifying

$$(1+m^2)\,x^2+2cmx+(c^2-a^2)=0 \qquad ...(3)$$

Let the points of intersection of the line with the circle be $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then $x_1$ and $x_2$ are the roots of equation (3).

$$\therefore\ x_1+x_2=-\frac{2cm}{1+m^2} \text{ and } x_1x_2=\frac{c^2-a^2}{1+m^2} \qquad ...(4)$$

Also $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on line (2)

$$\therefore\ y_1=mx_1+c$$

$$\text{and } y_2=mx_2+c$$

$\therefore$ by subtraction, we get

$$y_1-y_2=m(x_1-x_2) \qquad ...(5)$$

$$\text{Now } PQ^2=(x_1-x_2)^2+(y_1-y_2)^2$$

$$=(x_1-x_2)^2+m^2\,(x_1-x_2)^2 \qquad \text{from (5)}$$

$$=(1+m^2)(x_1-x_2)^2$$

$$=(1+m^2)\,[(x_1+x_2)^2-4x_1x_2]$$

$$=(1+m^2)\left[\frac{4c^2m^2}{(1+m^2)^2}-\frac{4(c^2-a^2)}{1+m^2}\right]$$

$$=\frac{4}{1+m^2}[c^2m^2-(c^2-a^2)\,(1+m^2)]$$

$$=\frac{4}{1+m^2}\left\{a^2(1+m^2)-c^2\right\}$$

$$\therefore\ PQ=\frac{2}{\sqrt{1+m^2}}\left\{a^2(1+m^2)-c^2\right\}^{\frac{1}{2}}$$

If $a^2(1+m^2)-c^2=0$, then $PQ=0$. In that case the line becomes a tangent to the circle.

## Exercise V (c)

1. Find the co-ordinates of the points where the line $3x-2y=0$ cuts the circle $x^2+y^2-3x+2y=1$ (*D.U. 1952*)

2. Find the co-ordinates of the points where the line $y=2x+1$ cuts the circle $x^2+y^2=2$ and find the length of the chord intercepted.

3. Show that the line $3x+4y+20=0$ touches the circle $x^2+y^2=16$ and find the point of contact. (*P.U.*)

4. Find the condition that the straight line $y=mx+c$ may touch the circle $(x-h)^2+(y-k)^2=a^2$. (*1939*)

5. Find the equations of the tangents to the circle $x^2+y^2=100$ which are parallel to the line $2x+y=0$.

6. Find the equation of the tangents to the circle $x^2+y^2=9$ which are perpendicular to the line $x-y-1=0$. (*P.U.*)

7. Find the condition that $lx+my+n=0$ may touch the circle $x^2+y^2=a^2$. Assuming the condition to have been satisfied, obtain the co-ordinates of the point of contact.

8. Find the condition that the line $x \cos \alpha+y \sin \alpha=p$ may touch the circle $x^2+y^2=a^2$. Find also the point of contact.

9. Find for what value of $k$ will the line $4x+3y+k=0$ touch the circle $2x^2+2y^2=5x$ ? (*P.U.*)

10. Find the condition that the circle $x^2+y^2+2gx+2fy+c=0$ may touch (*i*) the axis of $x$, (*ii*) the axis of $y$.

11. Show that the line $y=m(x-a)+a\sqrt{1+m^2}$ touches the circle $x^2+y^2=2ax$, whatever be the value of $m$.

12. Find the tangents to the circle $x^2+y^2=9$ which are inclined at an angle of 60° with the $x$-axis.

13. Prove that the line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y=12$. Find the point of contact.

14. Three tangents are drawn to the circle $x^2+y^2=25$ which form an equilateral triangle and one of them is parallel to $x$-axis. Find their equations. (*P.U. 1933*)

15. A circle touches the lines $x=0$, $y=0$ and $x=C$ ; find its equation.

**5·4. Position of a point w.r.t. a circle.**

Let there be a circle $x^2+y^2+2gx+2fy+c=0$ .. (1)
and a point $P(x_1, y_1)$ inside, on, or outside the circle. The point will be inside, on, or outside the circle according as its distance from the centre $(-g, -f)$ is $<, =$ or $>$ the radius.

*i.e.*, according as $(x_1+g)^2+(y_1+f)^2 <, =$ or $> g^2+f^2-c$

*i.e.*, according as $x_1^2+y_1^2+2gx_1+2fy_1+g^2+f^2 <, =$ or $> g^2+f^2-c$

*i.e.*, according as $x_1^2+y_1^2+2gx_1+2fy_1+c <, =$ or $> 0$.

This gives us the position of any point *w.r.t.* a circle.

*It may be noted that the expression on the L.H.S. is obtained by substituting the co-ordinates of the given point in the equation of the circle.*

**5·41. Tangents from a point.**

Let there be a circle $x^2+y^2=a^2$ and a point $(x_1, y_1)$.

The equation of any tangent to the circle is

$$y=mx+a\sqrt{1+m^2}.$$

The point $(x_1, y_1)$ will lie on the tangent if

$$y_1=mx_1+a\sqrt{1+m^2}$$

*i.e.*, if $y_1-mx_1=a\sqrt{1+m^2}$

or if $(y_1-mx_1)^2=a^2(1+m^2)$

or if $m^2(x_1^2-a^2)-2mx_1y_1+y_1^2-a^2=0$ ...(1)

This equation is a quadratic in $m$ and gives thereof, in general, two values of $m$, which are the slopes of the two tangents from $(x_1, y_1)$.

The two tangents are real and distinct, co-incident, or imaginary according as the roots of (1) are real, equal or imaginary.

*i.e.*, according as $4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2) >, =$ or $< 0$

*i.e.*, according as $x_1^2+y_1^2-a^2 >, =$ or $< 0$.

*i.e.*, according as the point $(x_1, y_1)$ lies outside, on or inside the circle.

**5·42.** *Locus of a point from which two perpendicular tangents can be drawn to a circle.*

Let the circle be $x^2+y^2=a^2$.

Take any point $P(x_1, y_1)$.

The equation of any tangent to the above circle is

$$y=mx+a\sqrt{1+m^2}.$$

This will pass through $(x_1, y_1)$ if

$$y_1=m_1x+a\sqrt{1+m^2}$$

or if $m^2(x_1^2-a^2)-2x_1y_1m+y_1^2-a^2=0.$

Let $m_1$, $m_2$ be the roots of this equation. Then $m_1$, $m_2$ ar the slopes of the two tangents from the point $(x_1, y_1)$.

The two tangents are cut perpendicular to each other.

$\therefore$ $$m_1m_2=-1$$

But $$m_1m_2=\frac{y_1^2-a^2}{x_1^2-a^2}$$

$\therefore$ $$\frac{y_1^2-a^2}{x_1^2-a^2}=-1$$

or $$x_1^2+y_1^2=2a^2;$$

Hence the locus of $P(x_1, y_1)$ is $x^2+y^2=2a^2$.

This is also a circle having the same centre.

**Example 1.** *Find the equations of the tangents from* (4, 2) *to the circle* $x^2+y^2=4$.

The equation of any tangent to the given circle is

$$y=mx\pm2\sqrt{1+m^2}$$

This passes through the point (4, 2) if

$$2=4m\pm2\sqrt{1+m^2}$$

*i.e.*, if $(2-4m)^2=4(1+m^2)$

or if $4+16m^2-16m=4+4m^2$

or if $12m^2-16m=0$ or if $m=0$, or $m=\frac{4}{3}$.

(*i*) When $m=0$, the two tangents are given by

$$y=\pm2.$$

The point (4, 2) lies on the tangent given by the +ve sign.

(*ii*) When $m=\frac{4}{3}$, the two tangents are given by

$$y=\tfrac{4}{3}x\pm2\sqrt{1+\tfrac{16}{9}}$$

or $$3y=4x\pm10.$$

The point (4, 2) lies on the tangent given by the −ve sign. Hence the two tangents from the point (4, 2) are

$$y=2 \text{ and } 4x-3y-10=0.$$

The above equation can also be found in another way. Any line through the point (4, 2) is given by

$$y-2=m(x-4)$$

or $$mx-y-4m+2=0 \qquad \ldots(1)$$

If this line is a tangent to the circle $x^2+y^2=4$, the perpendicular distance from the centre (0, 0) to the line must be equal to the radius of the circle.

*i.e.*, $$\frac{-4m+2}{\sqrt{1+m^2}}=2$$

or $$(-4m+2)^2=4(1+m^2)$$

or $$12m^2-16m=0 \quad i.e., \ m=0, \ \text{or} \ m=\tfrac{4}{3}$$

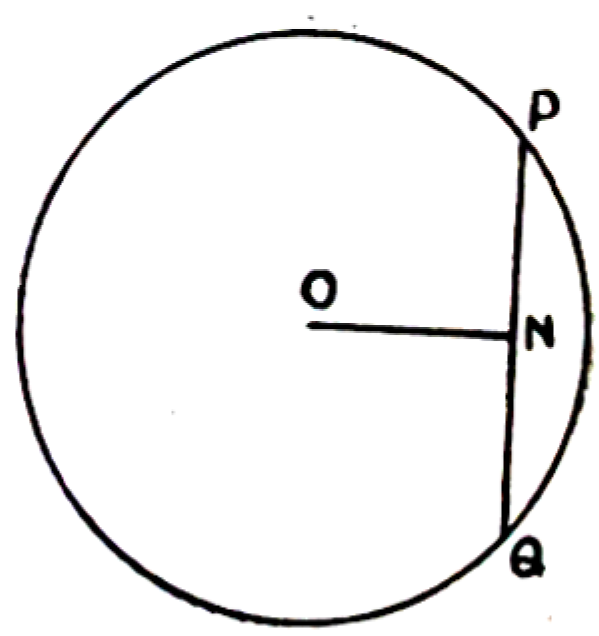
∴ The two tangents are

$$y-2=0$$

and $$y-2=\tfrac{4}{3}(x-4)$$

or $$4x-3y-10=0.$$

**5·43.** The equation of the chord of a circle when the mid point of the chord is known.




Let $N(h, k)$ be the mid-point. Let the equation of the circle be $x^2+y^2=a^2$.

Also Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the end points of the chord.

The equation of any line through $(h, h)$ is

$$y-k=m(x-k) \qquad \ldots(1)$$

where the value of $m$ is to be determined.

But $$m=\frac{y_1-y_2}{x_1-x_2}.$$

Also the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle

$\therefore \quad x_1^2+y_1^2=a^2$

and $\quad x_2^2+y_2^2=a^2$

$\therefore$ by subtraction $(x_1^2-x_2^2)+(y_1^2-y_2^2)=0$

or $$\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}.$$

And as N$(h, k)$ is the mid-point of P, Q

*i.e.*, $\quad x_1+x_2=2h$ and $y_1+y_2=2k$

$$\therefore \quad m=\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}=-\frac{h}{k}$$

$\therefore$ The reqd. equation is

$$y-k=\frac{-h}{k}(x-h)$$

or $\quad hx+ky=h^2+k^2.$

**Note**. *This method of finding the equation of a chord in terms of its mid-point is general and can be followed in the case of all second degree curves. But in the case of circle the equation can also be found by using the property that the line joining the mid-point of any chord to the centre is perpendicular to the chord. This fact will help us in finding the slope of the chord.*

**5·44. The locus of the mid-points of a system of parallel chords of a circle.**

Let the circle be $\quad x^2+y^2=a^2 \qquad \ldots(1)$

Also let PQ be one of the chords of the parallel system of the circle with N$(x_1, y_1)$ as its mid-point.

Let $m$ be the slope of the chords of the system.

Equation of PQ in terms of its mid-point $(x_1, y_1)$ is

$$xx_1+yy_1=x_1^2+y_1^2$$

$\therefore$ Slope of $\text{PQ}=-\dfrac{x_1}{y_1}$

But the slope is given to be $m$.

$$\therefore \quad -\frac{x_1}{y_1}=m \quad \text{or} \quad x_1+my_1=0$$

$\therefore$ The locus of $(x_1, y_1)$ is $x+my=0$.

This is a straight line passing through the centre (0, 0). It is also perpendicular to the system of parallel chords. Hence the locus of the mid-points of a system of parallel chords of a circle is the diameter perpendicular to the chords.

## Exercise V (D)

1. Find the equations of the tangents from

(*i*) the point $(-5, 0)$ to the circle whose radius is 3 and the centre is at the origin, (*1941*)

(*ii*) from (4, 5) to the circle $x^2+y^2-4x-2y+1=0$.

2. Find the equation of the chord of the circle $x^2+y^2=16$ whose mid-point is (3, 2).

3. Show that the mid-point of the chord $x \cos \alpha+y \sin \alpha=p$ of the circle $x^2+y^2=a^2$ is $(p \cos \alpha, p \sin \alpha)$.

4. Tangents are drawn from a variable point P to the circle $x^2+y^2=a^2$. If $\theta_1$, $\theta_2$ are the inclinations of these tangents, find the locus of P if,

(*i*) $\tan \theta_1+\tan \theta_2=k_1$

(*ii*) $\tan \theta_1 \cdot \tan \theta_2=k_2$

(*iii*) $\cot \theta_1+\cot \theta_2=k_3$.

5. Find the locus of a point, tangents from which to the circle $x^2+y^2=a^2$ are inclined at an angle $\alpha$.

## Chord of Contact. Pole and Polar

**5·5. Def.** *The* **chord of contact** *of the tangents from a point P to a circle is defined as the line joining the points of contact of the two tangents drawn from the point to the circle.*

It may be noted that as real and different tangents can be drawn only from a point outside the circle, a chord of contact is defined only in terms of a point outside the circle.

### 5·51. The equation of the chord of contact.

Let the circle be $x^2+y^2=a^2$ and a point $P(x', y')$ outside the circle.

Let $PT_1$ and $PT_2$ be the tangents to the circle from P, $(x_1, y_1)$ and $(x_2, y_2)$ the co-ordinates of $T_1$ and $T_2$ respectively.

Then $=n$ of $T_1P$, the tangent at $T_1$ is

$$xx_1+yy_1=a^2 \qquad ...(1)$$

and that of $T_2P$ is $xx_2+yy_2=a^2$ $\qquad ...(2)$

But both the tangents pass through P $(x', y')$.

$$\therefore \qquad x'x_1+y'y_1=a^2 \qquad ...(3)$$

and
$$x'x_2+y'y_2=a^2 \qquad ...(4)$$

Equations (3) and (4) show that the points $(x_1, y_1)$ and $(x_2, y_2)$ lie on the line whose equation is

$$xx'+yy'=a^2 \qquad ...(5)$$

This is the required equation of the chord of contact of P.

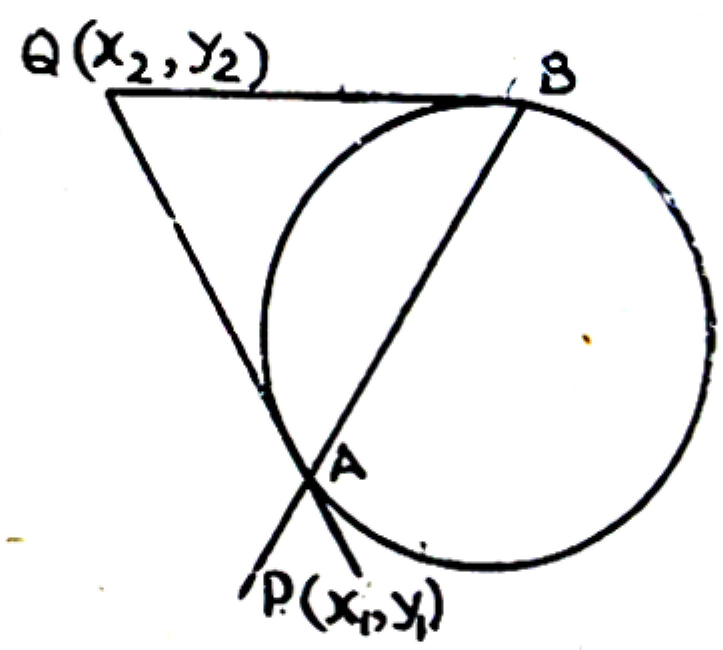
**Note 1.** The equation of the chord of contact of tangents from $(x', y')$ to the circle $x^2+y^2+2gx+2fy+c=0$, can be obtained in the same manner. The equation will be

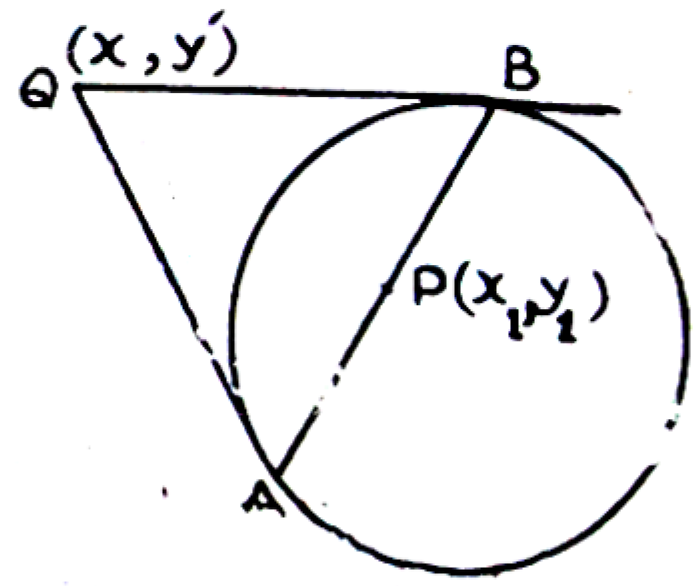
$$xx'+yy'+g(x+x')+f(y+y')+c=0.$$

**Note 2.** The equation of the chord of contact of the tangents from any point $(x', y')$ is of the same form as that of the equation of the tangents at $(x', y')$. Hence the equation of the chord of contact of the tangents is written down with the help of the same working rule as for the equation of the tangent.

**5·52. Def.** *The* **polar** *of a point P with respect to a circle is the locus of the point of intersection of tangents drawn at the extremities of chords through P. The point P is called the* **pole** *of the polar.*

### 5·53. The equation of the polar.




Let the circle be $x^2+y^2=a^2$ and $P(x_1, y_1)$ any point inside or outside the circle.

Let any chord through P meet the circle in A and B.

Also let Q $(x', y')$ be the point of intersection of the tangents at A and B.

Hence AB is the chord of contact of tangents from $Q(x', y')$ to the circle, and, therefore, its equation is

$$xx'+yy'=a^2.$$

But the point $P(x_1, y_1)$ lies on it,

$$\therefore \quad x_1x'+y_1y'=a^2$$

$\therefore$ The locus of $(x', y')$ is $xx_1+yy_1=a^2$.

Similarly it can be proved that the polar of $(x_1, y_1)$ w.r.t. the circle

$$x^2+y^2+2gx+2fy+c=0$$

*i.e.*, $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

**Note 1.** The form of the equation of the polar of a point is the same as that of the equation of the chord of contact and hence can be written down by the same working rule.

**Note 2.** If P lies on the circle, the polar of P will coincide with the tangent at P. If P lies outside the circle, the polar of P will coincide with the chord of contact of the tangents from P.

**5·54. The pole of a line.**

Let the circle be $\quad x^2+y^2=a^2 \quad$ ...(1)

and the line $\quad lx+my+n=0 \quad$ ...(2)

Let the co-ordinates of the pole of the line (2) be $(x_1, y_1)$.

The polar of $(x_1, y_1)$ w.r.t. the circle $x^2+y^2=a^2$ is

$$xx_1+yy_1=a^2$$

or $\quad xx_1+yy_1-a^2=0 \quad$ ...(3)

But (1) also represents the polar of $(x_1, y_1)$.

$\therefore$ equations (1) and (2) represent the same straight line.

$\therefore$ Comparing co-efficients, we get

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{-a^2}{n}$$

or $\quad x_1=-\dfrac{a^2l}{n}$ and $y_1=-\dfrac{a^2m}{n}$

Hence the pole is $\left(\dfrac{-a^2l}{n}, \dfrac{-a^2m}{n}\right)$.

### 5·55. Conjugate points and lines.

(i) *If the polar of a point P passes through another point Q, the polar of Q passes through P.*

Let the co-ordinates of P and Q be respectively $(x_1, y_1)$ and $(x_2, y_2)$.

The equation of the polar of P w.r.t. the circle $x^2+y^2=a^2$ is

$$xx_1+yy_1=a^2 \qquad \ldots(1)$$

The point $Q(x_2, y_2)$ lies on it,

$$\therefore \qquad x_1x_2+y_1y_2=a^2 \qquad \ldots(2)$$

Now the polar of Q is

$$xx_2+yy_2=a^2 \qquad \ldots(3)$$

On the basis of the condition (2), line (3) passes through $P(x_1, y_1)$, which proves the proposition.

These two points such that the polar of each passes through the other are called **conjugate points**.

(ii) *If the pole of a line lies on another, the pole of the second line lies on the first.*

Let the two lines be

$$l_1x+m_1y+n_1=0 \qquad \ldots(1)$$

and

$$l_2x+m_2y+n_2=0 \qquad \ldots(2)$$

The pole of (1) *w. r. t.* the circle $x^2+y^2=a_2$ is

$$\left(\frac{-a^2l_1}{n_1}, \frac{-a^2m_1}{n}\right)$$

and that of (2) is $\left(\dfrac{-a^2l_2}{n_2}, \dfrac{-a^2m_2}{n_2}\right)$

If the pole of (1) lies on (2), then

$$\frac{-a^2l_1l_2}{n_1}+\frac{-a^2m_1m_2}{n_1}+n_2=0$$

or

$$a^2l_1l_2+a^2m_1m_2-n_1n_2=0.$$

But equation (3) is the condition that the pole of (2) lies on (1).

These two lines such that the pole of each lies on the other are called **Conjugate lines.**

**5·56. Inverse points.** *If the line joining the centre O of a circle to a given point P meets the polar of P in Q, then P and Q are called* **inverse points** *w.r.t. the circle.*

It can be easily proved that OP. OQ=(radius)$^2$.

Let the circle be $x^2+y^2=a^2$ and the point P be $(x_1, y_1)$; Centre O is (0, 0)

$$\therefore \qquad \text{OP}=\sqrt{x_1^2+y_1^2}$$

Polar of P is $xx_1+yy_1=a^2$

As OP $\perp$ the polar of P

$$\therefore \qquad \text{OQ}=\text{perp. from O to the polar}=\frac{a^2}{\sqrt{x_1^2+y_1^2}}$$

$$\therefore \qquad \text{OP.OQ}=\sqrt{x_1^2+y_1^2}.\frac{a^2}{\sqrt{x_1^2+y_1^2}}=a^2=(\text{radius})^2.$$

## Exercise V (E)

1. Find the polar of the point

(*i*) (0, 6) w.r.t. the circle $x^2+y^2-4x-2y=4$

(*ii*) 2, $-7$ w.r.t. the circle $x^2+y^2=9$.

2. Find the pole of the line

(*i*) $x+y+3=0$ w.r.t. the circle $x^2+y^2+6x+8y+5=0$

(*ii*) $x \cos \alpha+y \sin \alpha=p$ w.r.t. $x^2+y^2=a^2$

3. Find the locus of the point whose polar w.r.t. the circle $x^2+y^2+2ax=0$ touches the circle $x^2+y^2=a^2$.

4. The chord of contact of tangents drawn from any point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that $a$, $b$, $c$ are in G. P. (*1950*)

5. Prove that the distances of two points from the centre of a circle are proportional to the distances of each from the polar of the other. [Salmon's Theorem]

6. If the polar of two points P, Q w.r.t. a circle meet in R, show that R is the pole of the line P Q.

7. The polar of a point w.r.t. the circle $x^2+y^2=a^2$ touches the circle $4(x^2+y^2)=a^2$. Show that the point lies on the circle $x^2+y^2=4a^2$.

8. Show that the polar of the origin w.r.t. the circle $x^2+y^2+2gx+2fy+c=0$ touches the circle $x^2+y^2=a^2$ if $c^2=a^2(f^2+g^2)$.

9. Prove that the locus of a point which moves in such a way that its distances from two fixed points are in constant ratio, is a circle. [This circle is known as circle of Appollonius].

CHAPTER VI

# THE CIRCLE (*Contd.*)

**6·1. Intersection of two circles.** The points of intersection of two circles are obtained by solving their equations simultaneously.

**Example.** *Find the points of intersection of the circles*

$$x^2+y^2-3x-5y-4=0 \qquad \ldots(1)$$

and

$$x^2+y^2-11x-11y+48=0 \qquad \ldots(2)$$

Subtracting (2) from (1)

$$8x+6y-52=0$$

or

$$4x+3y-26=0$$

or

$$y=\frac{26-4x}{3} \qquad \ldots(3)$$

Substituting the value of $y$ in (1);

$$x^2+\left(\frac{26-4x}{3}\right)^2-3x-5\left(\frac{26-4x}{3}\right)-4=0$$

or $$9x^2+(26-4x)^2-27x-15(26-4x)-36=0$$

or $$25x^2-175x+250=0$$

or $x^2-7x+10=0$ *i.e.*, $x=5$ or $2$.

Substituting the values of $x$ in (3), we get

$$y=2 \text{ when } x=5 \text{ and } y=6 \text{ when } x=2$$

$\therefore$ the points of intersection are (5, 2) and (2, 6).

**6·11.** Let $S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$

$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_2=0$$

be two given circles.

Consider the equation

$$S_1+\lambda S_2=0$$

$$i.e.,\ (1+\lambda)x^2+(1+\lambda)y^2+2(g_1+\lambda g_2)x+2(f_1+\lambda f_2)y+c_1+\lambda c_2=0.$$

This is an equation of the second degree in which

(i) co-efficient of $x^2$=co-efficient of $y^2$

and (ii) there is no term containing $xy$.

Hence this equation represents a circle for all values of $\lambda$.

Moreover the co-ordinates of a point that satisfy both $S_1=0$ and $S_2=0$, also satisfy $S_1+\lambda S_2=0$.

*Hence* $S_1+\lambda S_2=0$ *represents a circle through the common points of* $S_1=0$ *and* $S_2=0$ *whatever* $\lambda$ *be.*

**6·12.** It can similarly be shown that if $S=0$ is a circle, and $u=0$ is a straight line, then $S+\lambda u=0$ is a circle through the points of intersection of $S=0$ and $u=0$, for all values of $\lambda$.

**6·13. Common chord of two circles.** Let the equation of the two circles be written in the standard form as

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$S_2 \equiv x^2+y^2+2g_2x+2f_2y+c_3=0$$

Then $S_1-S_2=0$ *i.e.,* $2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0$, being an equation of the first degree represents a straight line. Moreover, it is satisfied by the co-ordinates of the common points of the two circles, and hence represents their common chord.

**Note 1.** The equations $S_1+\lambda S_2=0$ and $S_1+\lambda u=0$ contain only one parameter $\lambda$, a circle through the points of intersection of $S_1=0$ and $S_2=0$ or $S=0$ and $u=0$ is uniquely fixed if we are given one more condition about it.

**Note 2.** The general equation of a circle passing through the points of intersection of two circles $S_1=0$ and $S_2=0$ can also be written as $S_1+\lambda(S_1-S_2)=0$.

This method of writing the equation of a circle passing through the points of intersection of two circles sometimes saves tedius calculations.

**Example 1.** *Find the equation of the circle through the points of intersection of the circles*

$$x^2+y^2+2x+3y-7=0$$

*and*
$$x^2+y^2+3x-2y-1=0$$
*and through the point* $(1, 2)$. (1942)

The equation of any circle through the intersection of the given circles is

$$(x^2+y^2+2x+3y-7)+\lambda(x^2+y^2+3x-2y-1)=0.$$

This circle passes through $(1, 2)$.

$$\therefore \qquad (1+4+2+6-7)+\lambda(1+4+3-4-1)=0$$

or
$$\lambda=-2$$

$\therefore$ the required circle is

$$(x^2+y^2+2x+3y-7)-2(x^2+y^2+3x-2y-1)=0$$

or
$$x^2+y^2+4x-7y+5=0.$$

**Example 2.** *Find the circle which passes through the common points of*

$$x^2+y^2+3x+2y-14=0$$

*and*
$$x^2+y^2+4x-4y-4=0$$

*and has its centre on the line* $x+y=8$.

The equation of the common chord of the given circles is

$$x-6y+10=0.$$

$\therefore$ The equation of any circle through the common points of the given circles is

$$x^2+y^2+3x+2y-14+\lambda(x-6y+10)=0.$$

Centre of this circle is

$$\left(-\frac{3+\lambda}{2},\ -\frac{2-6\lambda}{2}\right).$$

If this lies on $x+y=8$,

$$-\frac{3+\lambda}{2}-\frac{2-6\lambda}{2}=8$$

or
$$-5+5\lambda=16 \quad \text{or} \quad \lambda=\tfrac{21}{5}$$

$\therefore$ The required circle is

$$x^2+y^2+3x+2y-14+\tfrac{21}{5}(x-6y+10)=0$$

or
$$5(x^2+y^2)+36x-116y+140=0.$$

**6·14. Touching circles.** *Two circles are said to touch each other if their two points of intersection become coincident. The common point is said to be their point of contact.*

We know that the point of contact is collinear with the centres of the two circles. *Thus two circles will touch, if the distance between their centres is equal to the sum or difference of their radii.* In the first case the contact is *external* while in the second case it is *internal*.

**Example.** *Show that the circles $x^2+y^2-2x-4y-3=0$ and $x^2+y^2-4x-6y+11=0$ touch. Find also the point sf contact.*

The two circles are

$$x^2+y^2-2x-4y-3=0 \qquad \ldots(1)$$

and

$$x^2+y^2-4x-6y+11=0 \qquad \ldots(2)$$

Subtracting (2) from (1)

$$2x+2y-14=0 \quad \text{or} \quad x+y-7=0$$

or

$$y=7-x \qquad \ldots(3)$$

Substituting this value of $y$ in (1)

$$x^2+(7-x)^2-2x-4(7-x)-3=0$$

or

$$2x^2-12x+18=0 \quad \text{or} \quad x^2-6x+9=0$$

which gives only one value of $x$ *i.e.*, 3.

$\therefore$ The two circles touch.

Substituting the value of $x$ in (3), we get $y=4$.

$\therefore$ The point of contact is (3, 4).

The two circles can also be found to touch each other by using the property stated above.

The centres of the two given circles are (1, 2) and (2, 3).

The radii of the two circles are $\sqrt{8}$ and $\sqrt{2}$.

$\therefore$ The distance between the two centres

$$=\sqrt{(2-1)^2+(3-2)^2}=\sqrt{2}$$

This distance is equal to the difference of the two radii.

Hence the given circles touch each other internally.

$\therefore$ The point of contact divides the segment between the two centres externally in the ratio $\sqrt{8}:\sqrt{2}$

$\therefore$ Its co-ordinates are $\left(\dfrac{\sqrt{8}\times 2-\sqrt{2}}{\sqrt{8}-\sqrt{2}},\ \dfrac{\sqrt{8}\cdot 3-2\sqrt{2}}{\sqrt{8}-\sqrt{2}}\right)$.

or

$$=(3, 4).$$

## Exercise VI (A)

1. Find the length of the common chord of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+bx+ay+c=0$.

*(D.H.S. 1948)*

2. Find the equation of the circle through the points of intersection of the circles

$$x^2+y^2+2x+3y-7=0$$

and

$$x^2+y^2+3x-2y-1=0$$

and through the point (1, 2). *(1942)*

3. Find the circle which passes through the common points of $x^2+y^2+3x+2y-14=0$ and $x^2+y^2+4x-4y=4$ and has its centre on the line $x+y=8$.

4. Show that the following circles touch

(*i*) $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ *(P.U.)*

(*ii*) $x^2+y^2+2x+2y+1=0$ and $x^2+y^2-4x-6y-3=0$. *(P.U.)*

5. The line $x \cos \alpha+y \sin \alpha=p$ cuts the circle $x^2+y^2=a^2$ in the points M and N. Show that the equation to the circle described on MN as diameter is

$$x^2+y^2-a^2=2p(x \cos \alpha+y \sin \alpha-p). \qquad (1949)$$

6. Show that the circles

$$x^2+y^2+2ax+c=0$$

and

$$x^2+y^2+2by+c=0$$

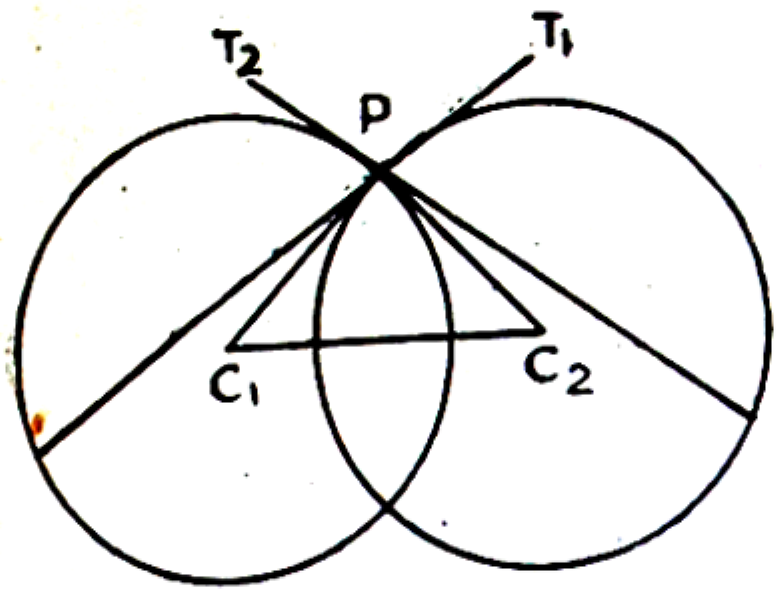
touch if $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c}$. *(1949)*

7. Find the equation of a circle through the points of intersection of $x^2+y^2=1$ and $x^2+y^2-2x-4y+1=0$ and touching the line $x+2y=0$.

## ORTHOGONAL CIRCLES

**6·2. Angle of intersection of two circles.** *The angle of intersection of two curves is defined as the angle between the tangents to the curves at the point of intersection.* Hence to find

the angle of intersection of two curves, we have to find the tangents at the points of intersection of the two curves and then find the angle between the two.




In the case of two circles, however, a simplified process can be followed. The tangents to the two circles are perpendicular to their respective radii and as such the angle between the tangents equals (or is supplementary to) the angle between the radii.

**6·21. Ths angle of intersection of the circles.**

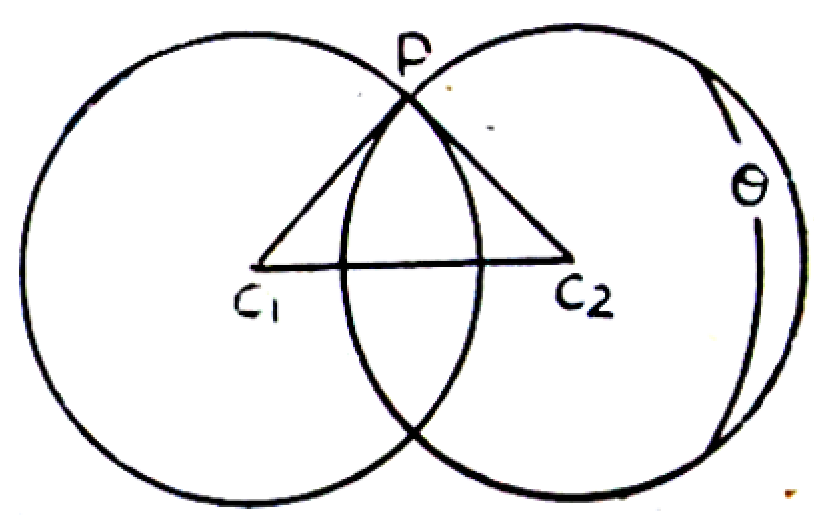
$$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad ...(1)$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad ...(2)$$

Let the circles (1) and (2) intersect in P and let $c_1$ and $c_2$ be their centres. Now the co-ordinates of $c_1$ and $c_2$ are

$$(-g_1, -f_1) \text{ and } (-g_2, -f_2)$$




Join $c_1c_2$, $c_1$P and $c_2$P.

Now $c_1Pc_2$ is the angle of intersection. Let it be equal to $\theta$.

Now $$\cos\theta=\frac{c_1P^2+c_2P^2-c_1c_2^2}{2c_1P.c_2P}$$

But $$c_1P^2=g_1^2+f_1^2-c_1$$
$$c_2P^2=g_2^2+f_2^2-c_2$$

and $$c_1c_2^2=(g_2-g_1)^2+(f_2-f_1)^2$$

$\therefore$ $$\cos\theta=\frac{g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2-(g_2-g_1)^2-(f_2-f_1)^2}{2r_1r_2}$$

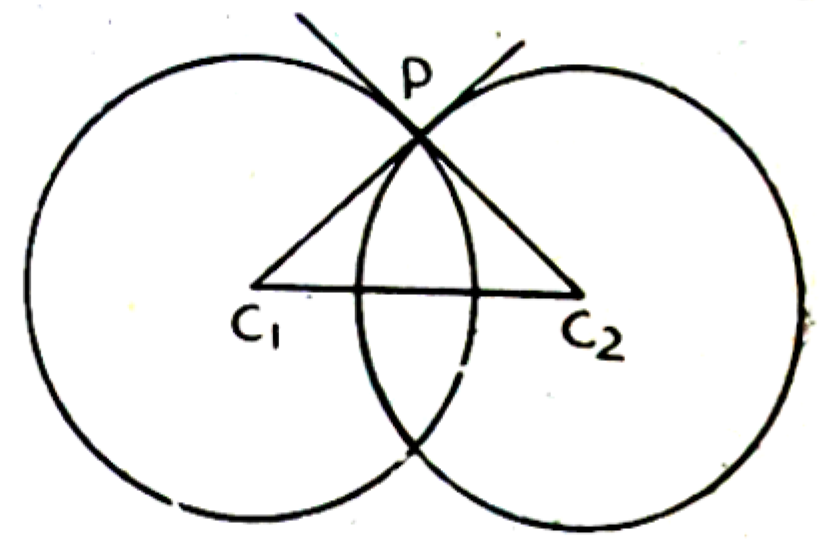
$$=\frac{2g_1g_2+2f_1f_2-c_1-c_2}{2r_1r_2}$$

where $r_1$ and $r_2$ are the radii of the two circles.

**Cor.** If $\theta=90$ $\cos\theta=0$ and therefore

$$2g_1g_2+2f_1f_2-c_1-c_2=0.$$

**6·22. Orthogonal circles.** Two circles are said to be **orthogonal** if they intersect at right angles.




In this case the angle between the two radii will also be a right angle and hence each radii will become the tangent of the other circle. Hence

$$PC_1^2+PC_2^2=C_1C_2^2$$

or $\quad r_1^2+r_2^2=d^2.$

**6·23.** On the basis of the above result we can find the condition that the circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and $\quad x^2+y^2+2g_2x+2f_2y+c_2=0$

may intersect orthogonally.

If $r_1$, $r_2$ be the radii of the two circles and $d$ the distance between the centres, then

$$r_1^2=g_1^2+f_1^2-c_1$$

$$r_2^2=g_2^2+f_2^2-c_2$$

and $\quad d^2=(g_1-g_2)^2+(f_1-f_2)^2$

The two circles are orthogonal if

$$d^2=r_1^2+r_2^2$$

*i.e.*, if, $\quad (g_1-g_2)^2+(f_1-f_2)^2=(g_1^2+f_1^2-c_1)+(g_2^2+f_2^2-c_2)$

or if $\quad 2g_1g_2+2f_1f_2=c_1+c_2$

This is the required condition.

**Example 1.** *Find the equation of the circle, centre $(3, -1)$ which cuts the circle $x^2+y^2=1$ orthogonally.*

The centre of the given circle is (0, 0) and its radius = 1.

The centre of the required circle is (3, −1).

Let the radius of the required circle be $r$.

As the two circles cut orthogonally,

$$r^2+1^2=3^2+1^2 \quad \text{or} \quad r=3$$

$\therefore$ The required circle is

$$(x-3)^2+(y+1)^2=9$$

or $\quad x^2+y^2-6x+2y+1=0$

**Example 2.** *Find the equation of the circle which cuts orthogonally the circles*

$$\left.\begin{aligned} &x^2+y^2-4x+2y+1=0, \\ &x^2+y^2+8x-6y+7=0 \\ \textit{and} \quad &x^2+y^2+6x-4y+9=0. \end{aligned}\right\} \text{Imp.}$$

The centre of the 1st circle is $(2, -1)$, and radius 2.

The centre of the 2nd circle is $(-4, 3)$, and radius $3\sqrt{2}$.

The centre of the 3rd circle is $(-3, 2)$, and radius 2.

Let the centre of the required circle be $(x_1, y_1)$ and its radius $r$.

$\because$ This circle is orthogonal to all the three given circles,

$\therefore$ $$(x_1-2)^2+(y_1+1)^2=4+r^2, \quad \ldots(1)$$

$$(x_1+4)^2+(y_1-3)^2=18+r^2, \quad \ldots(2)$$

and $$(x_1+3)^2+(y_1-2)^2=4+r^2 \quad \ldots(3)$$

Subtracting (1) from (2)

$$12x_1-8y_1+20=14$$

or $$6x_1-4y+3=0 \quad \ldots(4)$$

Also subtracting (3) from (2)

$$2x_1-2y_1+12=14$$

or $$x_1-y_1-1=0 \quad \ldots(5)$$

Solving (4) and (5) for $x_1$ and $y_1$, we get

$$\frac{x_1}{4+3}=\frac{y_1}{3+6}=\frac{1}{-6+4}$$

or $$x_1=-\tfrac{7}{2}, \quad y_1=-\tfrac{9}{2}$$

Substituting these values of $(x_1, y_1)$ in (1), we get

$$r^2=\tfrac{77}{2}$$

$\therefore$ the required circle is

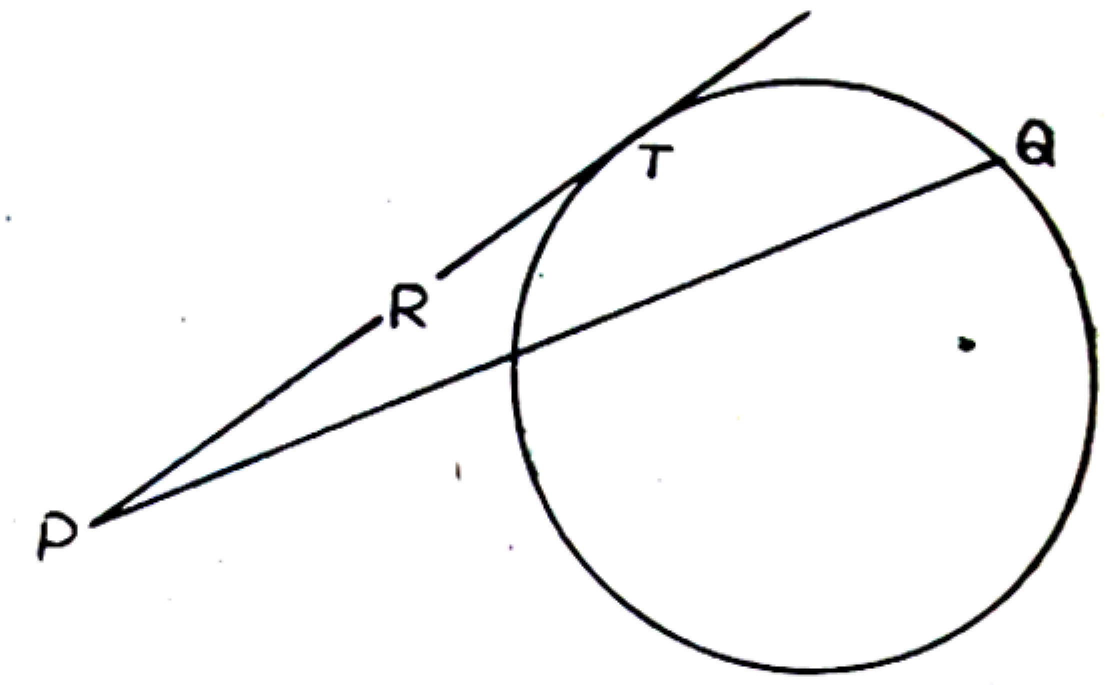
$$(x+\tfrac{7}{2})^2+(y+\tfrac{9}{2})^2=\tfrac{77}{2}$$

or $$x^2+y^2+7x+9y-6=0.$$

# RADICAL AXIS

## 6·3. Power of a point w.r.t. a circle.

Let there be a circle $x^2+y^2+2gx+2fy+c=0$ and a point $P(x_1, y_1)$. Take any line through P with inclination $\theta$. Then the equation of the line is




$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \quad \ldots(1)$$

*i.e.*, $\qquad x=x_1+r\cos\theta$

and $\qquad y=y_1+r\sin\theta$

for any point $(x, y)$ on the line at a distance $r$ from $(x_1, y_1)$.

If this point lies on the given circle,

then $\qquad (x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2+2g(x_1+r\cos\theta)$
$$+2f(y_1+r\sin\theta)+c=0$$

or $\qquad r^2+2\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}r$
$$+(x_1^2+y_1^2+2gx_1+2fy_1+c)=0.$$

The two roots of this quadratic in $r$ are the segments PR and PQ.

$$\therefore \quad PQ.PR=x_1^2+y_1^2+2g_1x+2f_1y+c.$$

This value of the product PQ.PR is independent of $\theta$, and hence is constant for any given point. This product is defined as the power of the point w.r.t. the given circle.

If Q and R coincide, (say at T), the secant PQR becomes tangent PT to the circle.

Then $\qquad PT^2=x_1^2+y_1^2+2gx_1+2fy_1+c$

which gives the length of a tangent to a circle from a given point.

**6·31. Radical Axis** *of two circles is defined as the locus of a point whose powers w.r.t. the two circles are equal.*

It may also be defined as the locus of a point tangent from which the two circles are equal.

**6·32.** Let there be two circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \ldots(1)$$
$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \ldots(2)$$

Let there be a point $P(x, y)$ on the radical axis of the two circles.

$\because$ P is a point on the radical axis of the two circles,

$\therefore$ power of P *w.r.t.* (1) = power of P *w.r.t.* (2)

*i.e.,* $x^2+y^2+2g_1x+2f_1y+c_1=x^2+y^2+2g_2x+2f_2y+c_2$

or $2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0,$

which is the equation of the radical axis. This equation is of first degree. Hence it represents a straight line.

**Note 1.** *The equation of the radical axis of two given circles is obtained by subtracting one equation from the other after making the co-efficients of $x^2$ and $y^2$ in the two equations equal.*

**Note 2.** *It may be observed that for two intersecting circles the radical axis is the common chord of the circles. In the case of touching circles the radical axis is the common tangent.*

**6·33. Radical centre of three circles.**

Let the equations of the three circles written in the standard form be $S_1=0$, $S_2=0$, $S_3=0$.

Then the equations of their radical axis, taken in pairs, are

$$S_1-S_2=0,\ S_2-S_3=0,\ S_3-S_1=0.$$

The third may be written as

$$(S_1-S_2)+(S_2-S_3)=0$$

which shows that it represents a line through the point of intersection of the first two.

Hence the three radical axes of three circles, taken in pairs, are concurrent. The point of concurrence is known as the **radical centre** of the three circles.

## Exercises VI (b)

1. Find the length of the tangent from (2, 5) to the circle
$$x^2+y^2-2x-3y-1=0. \qquad \textit{(D.H.S. 1946)}$$

2. Show that the following pairs of circle are orthogonal.

(i) $x^2+y^2-2ax+k^2=0$ and $x^2+y^2-2by-k^2=0$.

(ii) $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+4y-12=0$. *(1945)*

3. Find the equation of the circle which passes through the origin and cuts orthogonally each of the circles
$$x^2+y^2-8y+12=0 \text{ and } x^2+y^2-4x-6y-3=0.$$

4. Find the equation of a circle which cuts orthogonally the circles $x^2+y^2-4x+2y+1=0$, $x^2+y^2+8x-6y+7=0$ and $x^2+y^2+6x-4y+9=0$.

5. A circle cuts $x^2+y^2=4$ orthogonally and passes through (1, 3). Find the locus of its centre.

6. Find the radical axis of the following pairs of circles

(*i*) $3(x^2+y^2)+2x+y-4=0$ and $2(x^2+y^2)+10x-7y-100=0$

(*ii*) $x^2+y^2+ax+by+c=0$ and $x^2+y^2+bx+ay+c=0$. (*P.U.*)

(*iii*) $x^2+y^2+4x-4y+7=0$ and $x^2+y^2-6x+2y-3=0$.

7. Find the radical centre of the following circles

(*i*) $x^2+y^2+11x+5y+7=0$, $x^2+y^2+13x+6y+2=0$ and $x^2+y^2+17x+3y-3=0$. (*D.U. 1938*)

(*ii*) $x^2+y^2-1=0$, $x^2+y^2-4x+6y+2=0$ and

$$x^2+y^2-8x+12y+1=0$$ (*1946*)

8. Find the circle which cuts orthogonally the circles

$$x^2+y^2+3x+5y+7=0 \text{ and } x^2+y^2+x-y-1=0$$

and has its centre on the line $3x+2y+5=0$.

## Revision Exercise II

**1.** Find the equation of the circle which passes through the points $(a, 0)$, $(-a, 0)$ and $(0, b)$.

**2.** Write down the equations of the tangents to the circles $x^2+y^2=2ax$, $x^2+y^2=2by$ at the points of intersection and verify that they cut at right angles.

**3.** Show that the common chord of the circles

$$x^2+y^2-6x-4y+9=0 \text{ and } x^2+y^2-8x-6y+23=0$$

is a diameter of the latter circle.

**4.** Find the co-ordinates of the middle point of the chord $lx+my=1$ of the circle

$$x^2+y^2+2gx+2fy+c=0$$

**5.** Prove that the equation of the circle having for diameter the portion of the line

$$x \cos \alpha+y \sin \alpha=p$$

intercepted by the circle $x^2+y^2=a^2$ is

$$x^2+y^2-2p\,(x \cos \alpha+y \sin \alpha-p)-a^2=0$$

**6.** Prove that the equation of the circle circumscribing the triangle formed by the lines $x+y=6$, $2x+y=4$ and $x-2y=5$ is $x^2+y^2-17x-19y+50=0$

**7.** Find the equation of the circle whose diameter is the common chord of the circles

$$x^2+y^2+2x+3y+1=0 \text{ and}$$
$$x^2+y^2+4x+3y+2=0$$

**8.** Find the locus of a point which moves so that the sum of the squares of its distances from $n$ fixed points is constant.

**9.** Find the equation of the tangent to the circle $x^2+y^2-a^2$ at the point $(a \cos \theta, a \sin \theta)$ and show that the length of the tangent intercepted by the lines $x^2-y^2=0$ is $\pm 2a \sec 2\theta$.

**10.** Prove that the two circles which pass through the two points $(0, a)$ and $(0, -a)$ and touch the line $y=mx+c$, will cut orthogonally if $c^2=a^2\ (2+m^2)$.

**11.** Find the general equation of all circles any pair of which have the same radical axis as the circles

$$x^2+y^2=4 \text{ and } x^2+y^2+2x+4y=6$$

**12.** Prove that the polars of the point $(1, -2)$ with respect to the circles whose equations are

$$x^2+y^2+6y+5=0 \text{ and } x^2+y^2+2x+8y+5=0$$

coincide Prove also that there is another point the polars of which with respect to these circles are the same. Find its co-ordinates.

**13.** Prove that the polar of a given point with respect to any one of the circles $x^2+y^2-2kx+c^2=0$ where $k$ is variable, always passes through a fixed point, whatever be the value of $k$.

**14.** Tangents are drawn from the point $(h, k)$ to the circle $x^2+y^2=a^2$, prove that the area of the triangle formed by them and the straight line joining their points of contact is

$$\frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{(h^2+k^2)}$$

**15.** Find the equation to the circle whose centre is at the point $(\alpha, \beta)$ and which passes through the origin and prove that the equation of the tangent at the origin is

$$\alpha x + \beta y = 0$$

**16.** Two circles are drawn through the points $(a, 5a)$ and $(4a, a)$ to touch the axes of $y$. Prove that they intersect at an angle $\tan^{-1} \frac{40}{9}$.

**17.** If $y = mn$ be the equation of a chord of a circle whose radius is $a$, the origin of co-ordinates being one extremity of the chord and the axis of $x$ being a diameter of the circle. Prove that the equation of a circle of which this chord is the diameter is

$$(1+m^2)\ (x^2+y^2)-2a(x+my)=0$$

**18.** Find the value of $c$ in order that the two circles

$x^2+y^2+6x-8y+c=0$ and
$x^2+y^2=16$ may touch.

**19.** Show that two circles

$$x^2+y^2-4x-2y-4=0 \quad \text{and} \quad x^2+y^2-2x-2y-2=0$$

touch each other, and find their points of contact.

**20.** If the polar of a point P with respect to a circle with centre C, meets CP in P¹, then CP.CP¹ = (radius)².

**21.** Find the polar triangle of the circle $x^2+y^2=4$ whose one vertex is $(1, 1)$ and the second lies on $3x+y=0$.

**22.** The polar of the origin *w.r.t.* the circle

$x^2+y^2+2gx+2fy+c=0$ touches the circle
$x^2+y^2=a^2$ if $c^2=a^2\ (f^2+g^2)$

**23.** Show that the circles

$$x^2+y^2+2ax+c=0 \text{ and } x^2+y^2+2bx+c=0$$

touch if $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{c}$ *(P.U. 1942)*

## CHAPTER VII

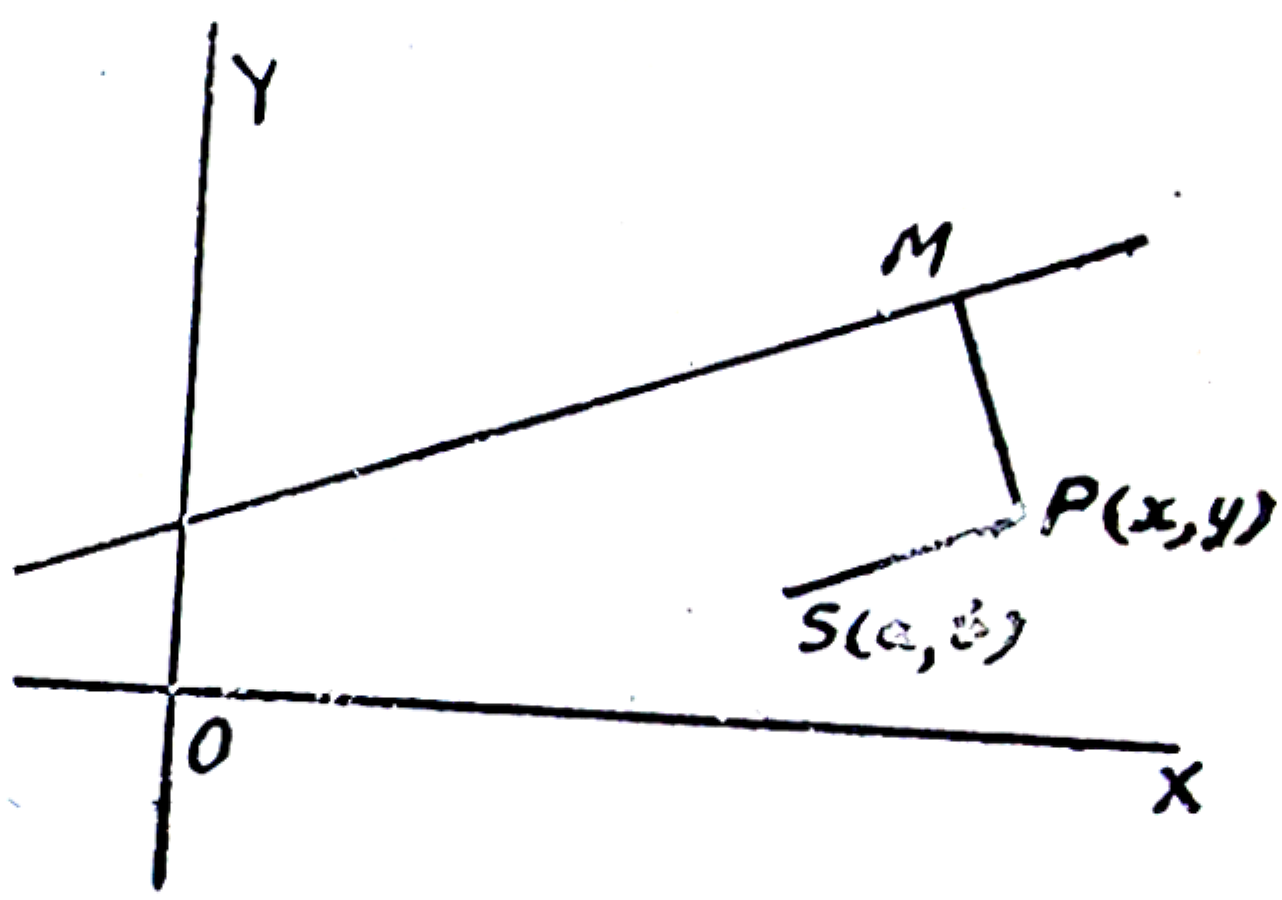
# PARABOLA

**7·1 Def.** *A* **parabola** *is the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line.*

The fixed point is called the focus of the parabola and the fixed line its **directrix.**

**7·11.** *To find the equation to a parabola with any point S (a, b) as focus and any line $lx+my+n=0$ as the directrix.*

Let P $(x, y)$ be any point on the parabola and PM the $\perp$ from P on the directrix.




Then by def., SP=PM ......$(i)$

Now SP$=\sqrt{(x-a)^2+(y-b)^2}$

PM$=\dfrac{lx+my+n}{\sqrt{l^2+m^2}}$

Substituting in $(i)$, we get

$$(l^2+m^2)\{(x-a)^2+(y-b)^2\}=(lx+my+n)^2 \qquad ......(ii)$$

is the required equation.

**Example.** *Find the equations to the parabolas whose focus and directrix are :—*

$(a)$ $(1, 1)$ ; $x-y+1=0$ $\qquad$ $(b)$ $(a, b)$; $\dfrac{x}{a}+\dfrac{y}{b}=1$

$(c)$ $(a, 0)$; $x+a=0$ $\qquad$ $(d)$ $(0, a)$; $y+a=0$

$(e)$ $(0, -a)$ ; $y=a$.

*Note.* Equation $(ii)$ of the last article may be rewritten as

$$(mx-ly)^2=2[a(l^2+m^2)+ln]x+2[b(l^2+m^2)+mn]y$$
$$+c^2-(l^2+m^2)(a^2+b^2) \qquad .....(iii)$$

We observe that

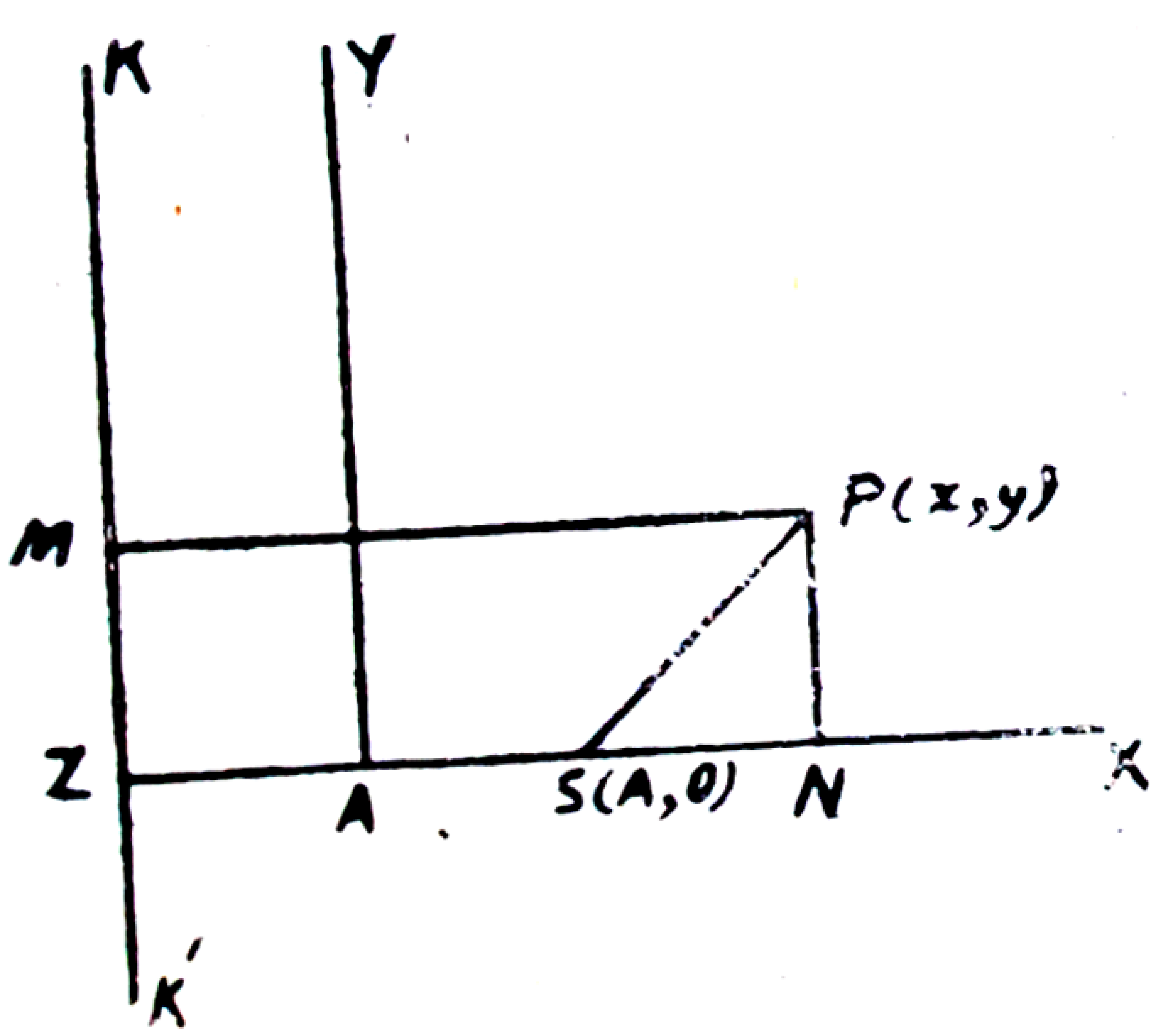
(1) The equation is of the 2nd degree in $x$, $y$.

(2) The terms of the 2nd degree in $x$, $y$ are forming a perfect square.

Thus the equation of any parabola is of the second degree in $x$, $y$ and the second degree terms form a perfect square. Conversely, though by no means, obviously, any equation possessing the characteristics (1) and (2) will represent a parabola.

**7·12.** We remarked in the first Chapter of this book that the methods of analytical geometry considerably simplify solutions of certain problems which would be long and difficult of solution by purely geometrical methods. By this time the student must have seen that the simplification depends upon the nature of equations that we employ. Now equations (*ii*) and (*iii*) of the last article are so cumbersome that it would be a job to find the co-ordinates of the common points of a line with the parabola, leave alone obtaining results which are less simple. Since all questions on the parabola would depend for solution, directly or indirectly, on the equation to the parabola, our first concern should, therefore, naturally be to obtain the same in as simple a form as possible. The choice of axes being ours, this we can do.

### 7·13. Equation to a parabola in a simplified form.




Let S be the focus and KK′ the directrix. Drop SZ ⊥ from S on KK′. Bisect it at A. Then since AS=AZ, the point A is by definition on the parabola.

We take A as the origin, ASX as the axis of $x$, AY the line through A ⊥ to AX, as the axis of $y$. Further we suppose SZ=$2a$

so that $ZA=AS=a$ and S is $(a, 0)$.

Let P $(x, y)$ be any point on the parabola so that $AN=x$, $NP=y$ and let PM be $\perp$ from P on KK$'$.

Then $SP=PM$ (by def.) $\quad \therefore SP^2=PM^2$ ......(1)

Also since $PM=ZN=ZA+AN=a+x$

$\therefore$ (1) becomes $(x-a)^2+y^2=(x+a)^2$ which reduces to $y^2=4ax$. ......(A)

(A) is the equation we sought.

**Cor. 1.** The focus S is $(a, 0)$ and the focal distance SP of $P(x, y)=PM=x+a$.

**Cor. 2.** Referred to parallel axes through $S(a, 0)$, the equation (A) becomes $y^2=4a(x+a)$.

Since Z is $(-a, 0)$, the equation (A), referred to parallel axes through Z, is $y^2=4a(x-a)$.

*Note :—A is known as the* **Vertex** *of the parabola.*

## Exercises VII (a)

1. Get the equations of Cor. 2 from first principles.

2. For what value of $a$ will the parabola $y^2=4ax$ pass through the points (*i*) $(3, -2)$ (*ii*) $(1, 1)$ ?

3. Find the point on the parabola $y^2=4ax$ whose ordinate is equal to its abscissa.

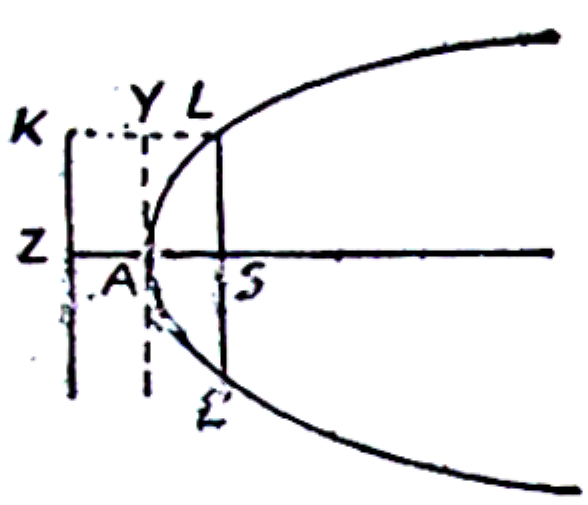
4. Find the ordinate of a point or points on the parabola $y^2=4ax$ whose abscissa is equal to $a$.

5. From first principles or otherwise, show that the equation to the parabola, taking A as origin and AS as the axis of $y$ is $x^2=4ay$.

[*Otherwise* :—Interchange $x$ and $y$ in $y^2=4ax$].

6. Show that, whatever $m$ or $t$, the points whose co-ordinates are $(am^2, -2am)$ ; $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ and $(at^2, 2at)$ all lie on the parabola $y^2=4ax$.

**7·14.** *To trace the parabola $y^2=4ax$.*




1. (*i*) Putting $y=0$, we get $x=0$ so that $x$-axis meets the parabola at A(0, 0). [Also see Note Art 7·25.]

(*ii*) Putting $x=0$, $y^2=0$ so that $y$-axis meets the parabola in two coincident points at A. The $y$-axis is, therefore, a tangent to the parabola at the vertex A. [Also see Art. 7·22].

2. If $x$ is negative; $y^2$ becomes negative and therefore $y$ becomes imaginary, so that no part of the curve lies on the negative side of the $y$-axis (or to the left of A).

3. When $x$ is positive, there are two values of $y$, equal in magnitude but opposite in sign. This means that all chords of the curve $\perp$ to the axis of $x$ are bisected by it and the portions of the curve above and below the axis of $x$ are in all respects equal. We express this by saying that the parabola is symmetrical, with respect to the $x$-axis.

4. As $x$ increases, $y$ also increases and there is no limit to this increase of $x$ and $y$. The curve, therefore, extends to infinity on the right side of the $y$-axis both below as well as above the $x$-axis.

The form of a parabola is therefore as given in the figure.

**7·15 Defs.** (1) The line SZ through S $\perp$ to the directrix which is also the line of symmetry for the parabola is called its axis.

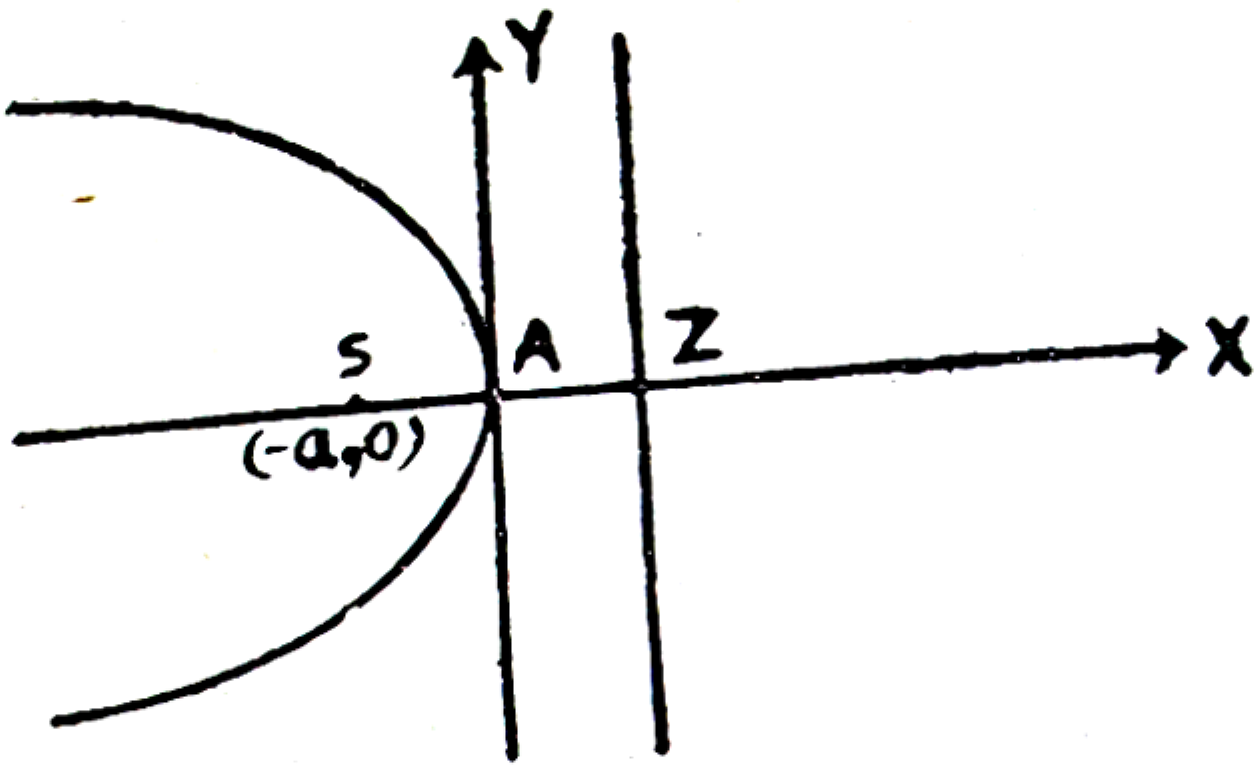
(2) The chord LSL′ through S and $\perp$ to the axis is called the latus rectum. [Fig. Art. 7·14.]

**7·16. Length of the latus rectum.**

Since SL=LK=ZS=$2a$ and LL′=2SL ($x$ axis bisects all chords $\perp$ to it).

$\therefore$ length of the latus rectum$=4a$.

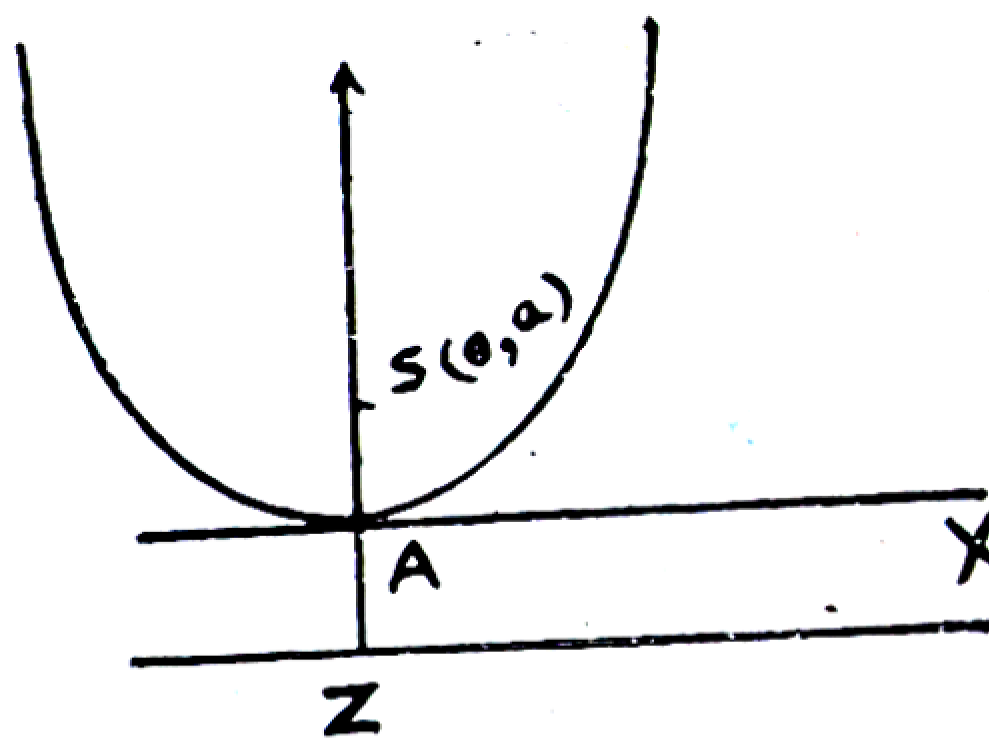
The following example is important :—




**Example.** *Trace the parabolas* (*i*) $y^2 = -4ax$, (*ii*) $x^2 = 4ay$, (*iii*) $x^2 = -4ay$.

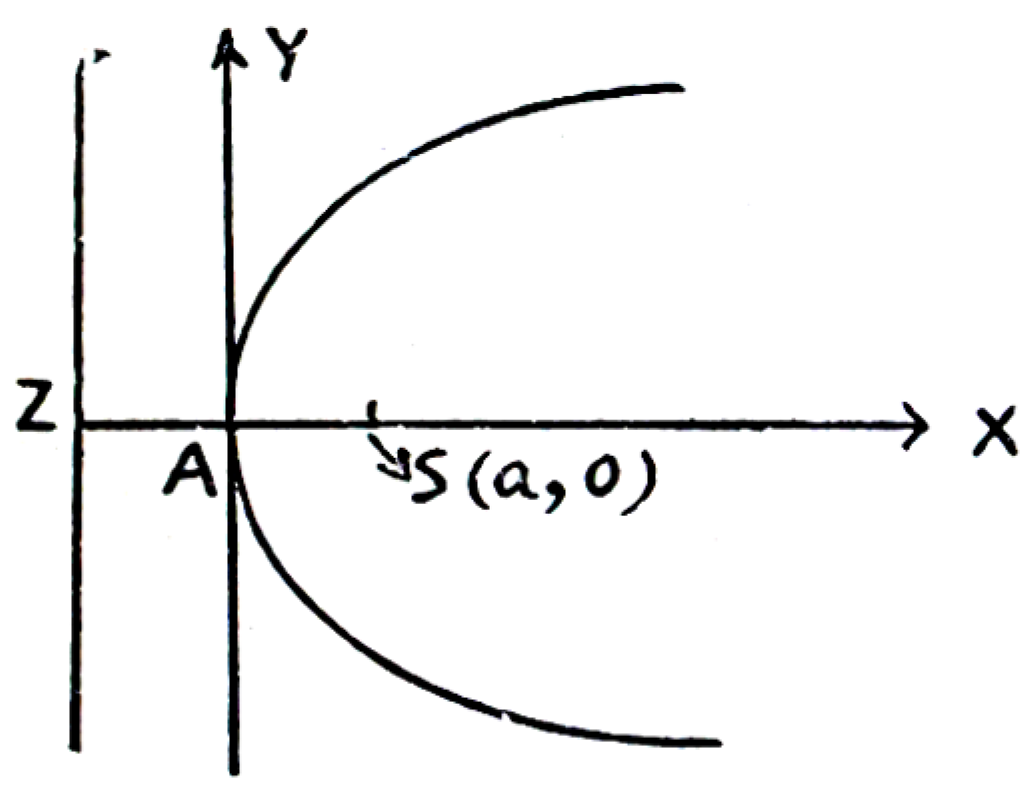
(*i*) For positive values of $x$, $y$ becomes imaginary, so that there is no part of the curve to the right of A. The rest, 7·15 (See Fig.)

(*ii*) 1. For $y=0$, $x^2=0$, so that the $x$-axis is a tangent to the curve at (0, 0).




2. If $y$ is negative, $x$ becomes imaginary ∴ no part of the curve lies below the $x$-axis.

3. For a positive $y$, there are two equal and opposite values of $x$, giving symmetry about the $y$-axis.




4. As $y$ increases $x$ **increases** so that the curve extends **to infinity** on both sides of the $y$-axis. See Fig.

(*iii*) For a positive value of $y$, $x$ becomes imaginary.

∴ the curve lies entirely below the $x$-axis. The rest as in (*ii*). See Fig.

## Exercises VII (b)

1. Find the axis, the vertex, the length and the equation of the latus rectum of the parabola whose focus is S(−1, 1) and the directrix $x+y+1=0$.

The axis is a line through $(-1, 1) \perp$ to $x+y+1=0$ and is, therefore, given by $x+1-(y-1)=0$ or $x-y+2=0$. Also $x+y+1=0$ and $x-y+2=0$ solved together give $Z(-\frac{3}{2}, \frac{1}{2})$ and S being $(-1, 1)$, A the mid-point of SZ is $(-\frac{5}{4}, \frac{3}{4})$.

$$\text{Also } a=AS=\sqrt{\frac{1}{16}+\frac{1}{16}}=\frac{\sqrt{2}}{4}$$

$\therefore$ the length of the latus rectum $=4a=\sqrt{2}$.

Latus rectum is a line through $S \perp$ to the axis.

$\therefore$ its equation is $(x-1)+(y+1)=0$ *i.e.*, $x+y=0$.

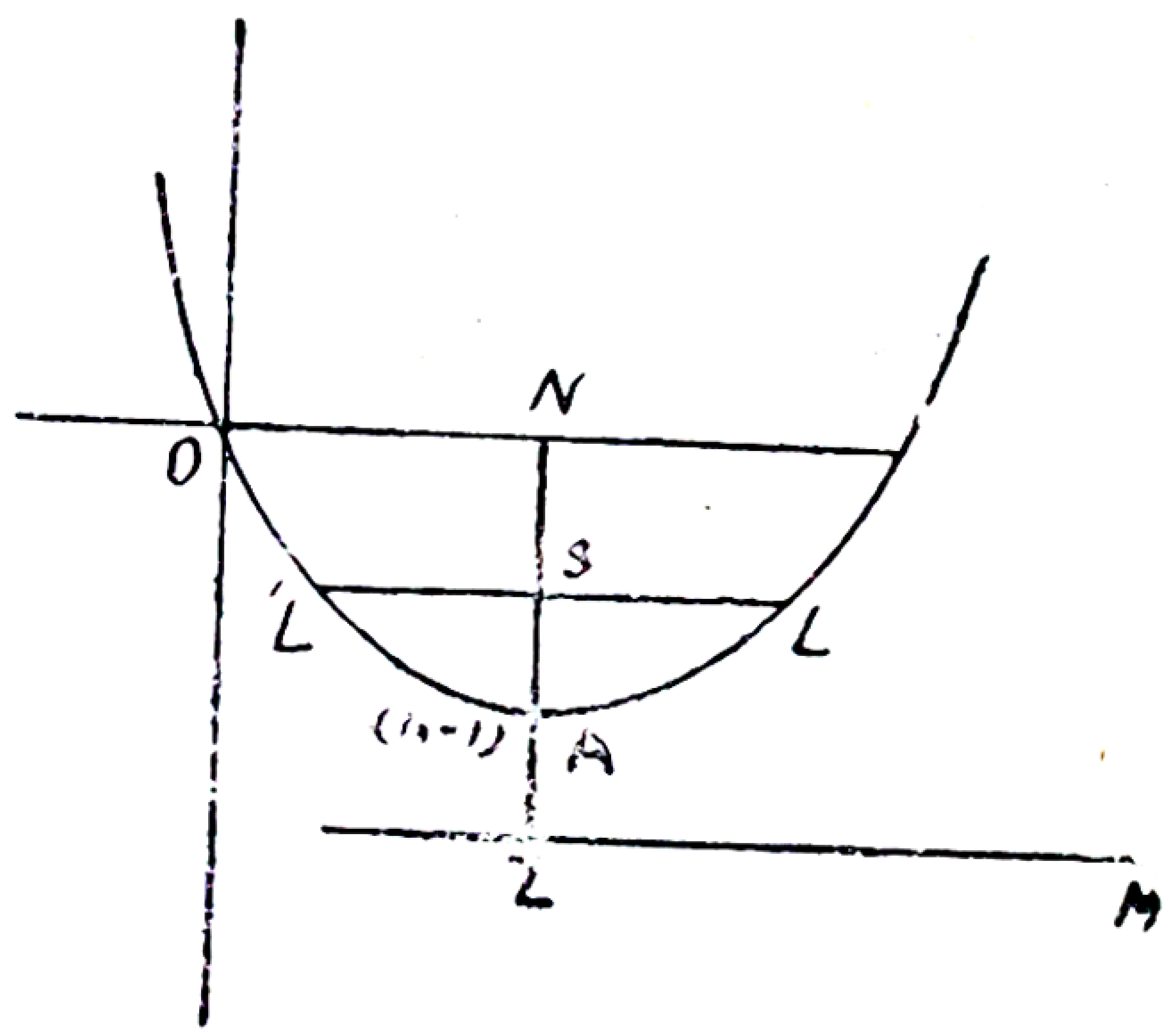
2. Find the axis, the vertex, the length and equation of the latus rectum of the parabola whose

(*i*) focus is $(-1, 2)$ and directrix $x-2y=15$.

(*ii*) focus is $(2, 3)$ and directrix $x-2y=6$.

3. Find the vertex, focus, the axis and directrix of the parabola $x^2-2x-y=0$.

Writing it as $(x-1)^2=y+1$, we see that if we change the origin to $(1, -1)$, the equation becomes $x^2=y$, a parabola whose latus rectum is unity.




With reference to the old origin, therefore, the vertex A is $(1, -1)$, the focus S is $(1, -\frac{3}{4})$. The axis AS is $x=1$ and the directrix KZ is $y=-\frac{5}{4}$.

4. Find the vertex, focus, axis and directrix for the parabolas whose equations are

(*a*) $y^2=4x+4y$ (*b*) $x^2-2x-12y-11=0$.

[**Hint.** Put (*a*) as $(y-2)^2=4(x+1)$ and (*b*) as $(x-1)^2=12(y+1)$ etc.]

**7·17. Parametric Equations.** In question 6 Ex. VII (a) we saw that the points $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ and $(at^2, 2at)$ all lie on

the parabola $y^2=4ax$ whatever values $m$ and $t$ may have. By varying $t$ or $m$, we have different points on the parabola and giving them all possible values, we get all possible points on the parabola. Thus the equations $x=at^2$, $y=2at$ or

$x=\dfrac{a}{m^2}$, $y=\dfrac{2a}{m}$ could be looked upon as the equations generating the parabola and are, therefore, called the parametric equations of the parabola $y^2=4ax$, $t$ or $m$ being the parameters.

In questions, it would help us considerably to take any one of the two forms as representing a point on a parabola. We shall often refer the point $(at^2, 2at)$ briefly as the point '$t$' and the point $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ as the point '$m$' on the parabola.

**Example 1.** *What are the values of t for the vertex and the ends of the latus rectum of the parabola* $y^2=4ax$ ?

**Example 2.** *What are the values of t for the points of intersection of the line* $4x-3y=4a$, *with the parabola* $y^2=4ax$ ?

Also find the co-ordinates of the points of intersection.

The point $(at^2, 2at)$ is on the parabola for all $t$. For it to lie on $4x-3y=4a$ we must have $4at^2-6at=4a$ or $2t^2-3t-2=0$ giving $t=2$ and $-\frac{1}{2}$ which are the values required.

Substituting for $t$ in $(at^2, 2at)$ we get $(4a, 4a)$ and $\left(\dfrac{a}{4}, -a\right)$ as the points of intersection.

**Example 3.** *Find t for the common points of* $y^2=4x$ *and* $x=y-1$. *Also find the points of intersection.*

**7·18.** *To find the equation of the chord of the parabola* $y^2=4ax$ *joining the points* $(at_1^2, 2at_1)$ *and* $(at_2^2, 2at_2^2)$.

This equation is $\dfrac{y-2at_1}{2a(t_1-t_2)}=\dfrac{x-at_1^2}{a(t_1^2-t_2^2)}$

or $\dfrac{y-2at_1}{2}=\dfrac{x-at_1^2}{t_1+t_2}$ or $(t_1+t_2)y-2at_1(t_1+t_2)=2x-2at_1^2$

or $(t_1+t_2)y=2(x+at_1t_2)$

This equation is *important.*

**Example 4**. *Deduce from the above that if the chord passes through the focus, $t_1t_2=-1$.*

[**Hint** :—$(a, 0)$ must satisfy $(i)$]

**7·19. Geometric interpretation of $y^2=4ax$.** ...(1)

If P be $(x, y)$, we have from the figure (Art. 7·13)

$$NP^2=4AS.\ AN \quad ...(2)$$

which may be stated thus :—

*The ordinate of any point on the parabola is a mean proportional between its abscissa and the latus rectum of the parabola.*

Equation (2) also expresses that P moves in such a manner that the square of its distance from a fixed line AX, varies as its distance from a line AY which is perpendicular to it.

*Conversely.* If a point P moves under this condition then it describes a parabola for which the first line will be the axis, the second, the tangent at the vertex, and the constant of variation, the length of the latus rectum.

**Example**. *The parabola of latus rectum 4 units having $3x+4y-4=0$ for its axis and $4x-3p+7=0$ for the tangent at the vertex is given by*

$$\left(\frac{3x+4y-4}{\sqrt{3^2+4^2}}\right)^2=4\left(\frac{4x-3y+7}{\sqrt{4^2+3^2}}\right)$$

or $(3x+4y-4)^2=20(4x-3y+7)$.

**7·2. Points of intersection of a straight line with a parabola**.

Let the line be $y=mx+c$ ...(1)

and the parabola, $y^2=4ax$ ...(2)

The common points, as usual, are obtained by solving (1) and (2) simultaneously. Substituting for $y$ from (1) in (2), we get $(mx+c)^2=4ax$

or $m^2x^2+2(mc-2a)\ x+c^2=0$. ...(3)

Equation (3) is a quadratic in $x$ which will give us the abscissae of the two points of intersection. The ordinates may then be found from (1).

Thus every straight line meets a parabola in two points which would be real and distinct, coincident, or imaginary according as (3) has real and distinct, equal, or imaginary roots.

**Cor. Condition of tangency of y=mx+c and the parabola y²=4ax.**

The line will be a tangent to the parabola if roots of (3) above are equal, *i.e.*, if $(mc-2a)^2-m^2c^2=0$

or $$4amc-4a^2=0$$

whence $c=\dfrac{a}{m}$ *which is the condition required.*

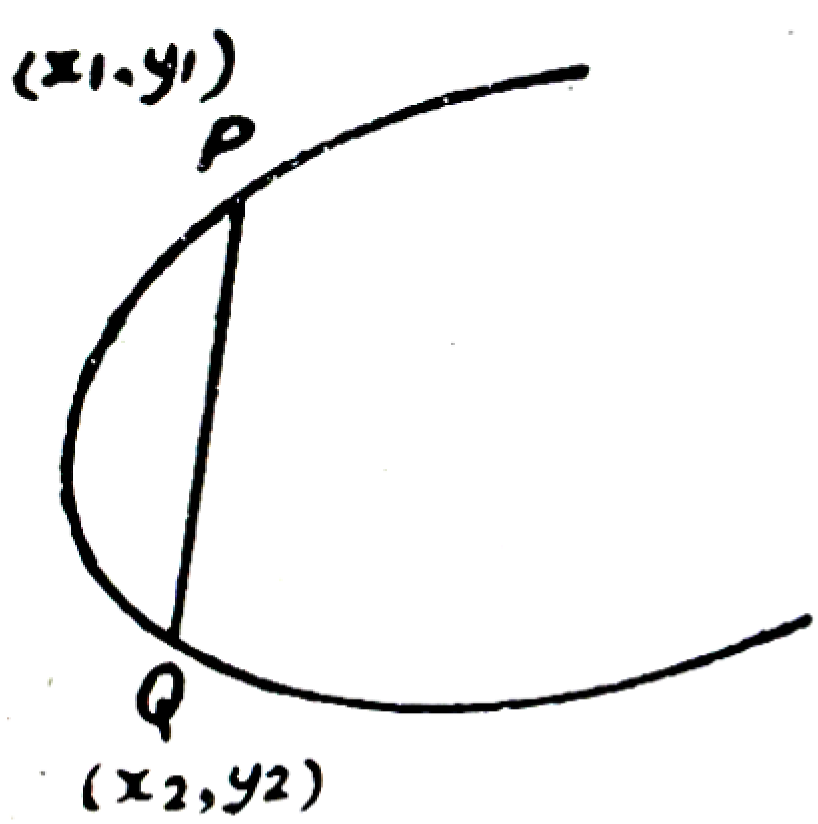
Thus the line $y=mx+\dfrac{a}{m}$ ...(A)

will be a tangent to the parabola (2) whatever $m$ may be.

*Equation (A) is important.*

**7·21. Length of the chord intercepted by the parabola y²=4ax on the st. line y=mx+c.**

If $P\ (x_1,\ y_1)$ and $Q\ (x_2,\ y_2)$ be the ends of the chord, we have, since, P, Q, lie on $y=mx+c$.




$$y_1=mx_1+c,\ y_2=mx_2+c\ ;$$

so that $y_1-y_2=m(x_1-x_2)$ ...(1)

Now $PQ^2=(x_1-x_2)^2+(y_1-y_2)^2$
$=(x_1-x_2)^2(1+m^2)$ by (1)

Now $x_1$, $x_2$ are the roots of
$$m^2x^2+2(mc-2a)x+c^2=0$$

$$\therefore\quad x_1+x_2=-\frac{2(mc-2a)}{m^2}\text{ and } x_1x_2=\frac{c^2}{m^2}$$

giving $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$
$$\frac{4(mc-2a)^2-4m^2c^2}{m^4}=\frac{16a(a-mc)}{m^4}.$$

$$\therefore \quad PQ=\sqrt{1+m^2}\,(x_1-x_2)$$

$$=\frac{4}{m^2}\sqrt{a(a-mc)(1+m)^2}.$$

**Cor.** We could deduce the condition of tangency from the expression for PQ as well. The line would be a tangent when PQ=0 which gives $a-mc=0$

or $\quad c=\dfrac{a}{m}$ as before.

## Secant, Tangent and Normal

### 7·22. (a) Chord joining two points on a parabola.

Let the points be P $(x_1, y_1)$ and Q $(x_2, y_2)$ and the parabola, $\quad y^2=4ax.$ ......(1)

Since P $(x_1, y_1)$ and Q $(x_2, y_2)$ are on the parabola,

$y_1^2-4ax_1=0$ ...(2) $\qquad y_2^2-4ax_2=0$ ......(3)

We have from (2) and (3)

$$(y_1^2-y_2^2)-4a(x_1-x_2)=0 \text{ or } \frac{y_1-y_2}{x_1-x_2}=\frac{4a}{y_1+y_2} \quad \text{......(4)}$$

Equation to PQ is $\qquad y-y_1=\dfrac{y_1-y_2}{x_1-x_2}(x-x_1)$

which with the help of (4) becomes

$$y-y_1=\frac{4a}{y_1+y_2}(x-x_1) \quad \text{......(5)}$$

### (b) Tangent at P$(x_1, y_1)$.

or $\quad y(y_1+y_2)-y_1^2-y_1y_2=4ax-4ax_1$

$\qquad y(y_1+y_2)=4ax+y_1y_2$ by (2) ...(A)

*Equation (A) is important and we shall refer to it more than once subsequently.*

**Cor.** The chord joining the points.

$(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is easily seen to be

$$(t_1+t_2)y=2(x+at_1t_2)$$

The secant PQ would become a tangent at P if Q coincides with P. Therefore making $y_2=y_1$ in (6), we have for the equation of the tangent $y-y_1=\dfrac{4a}{2y_1}(x-x_1)$

or $\quad yy_1-y_1^2=2ax-2ax_1$

which with the help of (2) reduces to

$$yy_1=2a(x+x_1) \qquad \ldots\ldots(B)$$

**Cor 1.** Tangent at the vertex A(0, 0) is $x=0$, *viz.*, the $y$ axis.

**Cor. 2.** Tangent at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ is $y=mx+\dfrac{a}{m}$.

**Cor 3.** Tangent at $(at^2, 2at)$ is $ty=x+at^2$

(*c*) **Normal at P(x₁, y₁).**

The normal at $P(x_1, y_1)$ is a line through P and $\perp$ to the tangent $yy_1-2a(x+x_1)$ and is $\therefore$ given by

$$y-y_1=-\frac{y_1}{2a}(x-x_1) \qquad \ldots\ldots(C)$$

**Cor. 1.** Normal at $(am^2, -2am)$ is given by (C) in the form

$$y=mx-2am-am^3.$$

**Cor. 2.** Normal at $(at^2, 2at)$ is $tx+y=2at+at^3$.

*Note 1.* All the equations obtained in this article are extremely important.

*Note 2. The Geometric Interpretation of the various parameters.*

The student can see for himself now that the parameter '$m$', in $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ is the slope of the tangent at the point and '$t$' in $(at^2, 2at)$, the reciprocal of the same, and '$m$' in $(am^2, -2am)$, the slope of the normal at the point.

*Note 3.* The equations of Cor. 2, Art. 7·22 (*b*) and Cor. 1, Art 7·21 (C) may also be obtained otherwise.

(*i*) Write (B) as $y=\dfrac{2a}{y_1}(x+x_1)$ and put $\dfrac{2a}{y_1}=m$ so that

$$y_1=\frac{2}{m}, \text{ and } \therefore x_1=\frac{y_1^2}{4a}=\frac{4a^2}{4am^2}=\frac{a}{m^2}.$$

Equation (B) then reduces to

$$y=m\left(x+\frac{a}{m^2}\right)=mx+\frac{a}{m} \text{ as before.}$$

The point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

(*ii*) Making $\dfrac{-y_1}{2a} = m$ in (C) above, we get $y_1 = -2am$

and $x_1 = \dfrac{y_1^2}{4a} = \dfrac{4a^2m^2}{4a} = am^2$.

Substitution in (C) gives $y + 2am = m(x - am^2)$

or $y = mx - 2am - am^3$

The foot of the normal is $(am^2, -2am)$.

These equations are sometimes spoken of as the equations of the tangent and the normal **in terms of their slopes.**

**7·23. The point of contact.** If a line is given to be a tangent to a parabola, to find the pt. of contact we solve the equations of the line and the parabola simultaneously and must get a perfect square in one of the variables. That would give us one of the co-ordinates of the pt. of contact and the other could be found from the equation of the st. line.

**Example. 1.** *Prove that the line $4x - 2y + 1 = 0$ touches the parabola $y^2 = 4x$. Also find the point of contact.*

Eliminating $x$ between two equations

$$y^2 = 2y - 1 \text{ or } (y-1)^2 = 0$$

The two roots of this equation are equal. Hence the line intersects the parabola in two coincident points *i.e.*, the line touches the parabola.

Putting $y = 1$, in the equation of the line, $x = \frac{1}{4}$.

$\therefore$ the point of contact is $(\frac{1}{4}, 1)$

The *method* illustrated in the following example, however, should be preferred, especially in equations with algebraic co-efficients.

**Example 2.** *Find the condition that $lx + my + n = 0$ should touch $y^2 = 4ax$. Also find the point of contact.*

**Let** $\quad lx + my + n = 0 \qquad \ldots\ldots(1)$

touch $y^2 = 4ax \qquad \ldots\ldots(2)$

at $(x_1, y_1)$

Then (1) should be the same as

$$yy_1=2a(x+x_1) \quad \text{or } 2ax-yy_1+2ax_1=0 \qquad ...(3)$$

Comparing co-efficients, we have

$$\frac{2a}{l}=\frac{-y_1}{m}=\frac{2ax_1}{n} \text{ whence } x_1=\frac{n}{l},\ y_1=\frac{-2am}{l}.$$

Also $(x_1, y_1)$ is on (2) $\therefore$ $y_1^2=4ax_1$.

Substituting for $x_1, y_1$, we get $\dfrac{4a^2m^2}{l^2}=\dfrac{4an}{l}$

or $am^2=nl$ as the condition required.

And the point of contact is $\left(\dfrac{n}{l},\ \dfrac{-2am}{l}\right)$

**7·24. The equation of a chord of a parabola in terms of the co-ordinates of its mid-point.**

Let PQ be the chord whose mid. pt. is $M(h, k)$.

Let P be $(x_1, y_1)$ and Q, $(x_2, y_2)$.

$$\text{Then } \left.\begin{aligned} 2h&=x_1+x_2 \\ 2k&=y_1+y_2 \end{aligned}\right\} \qquad ...(1)$$

If '$m$' be the slope of PQ, its equation is

$$y-k=m(x-h) \qquad ...(2)$$

Also the slope of $PQ=\dfrac{y_1-y_2}{x_1-x_2}$

Points $(x_1, y_1)$ and $(x_2, y_2)$ lie on the parabola

$$\therefore \qquad y_1^2=4ax_1 \qquad ...(3)$$

$$\text{and} \qquad y_2^2=4ax_2 \qquad ...(4)$$

Subtracting (4) from (3)

$$y_1^2-y_2^2=4ax_1-4ax_2$$

or $(y_1-y_2)\ (y_1+y_2)=4a(x_1-x_2)$

$$i.e., \quad \frac{y_1-y_2}{x_1-x_2}=m=\frac{4a}{y_1+y_2}=\frac{4a}{2k}=\frac{2a}{k}$$

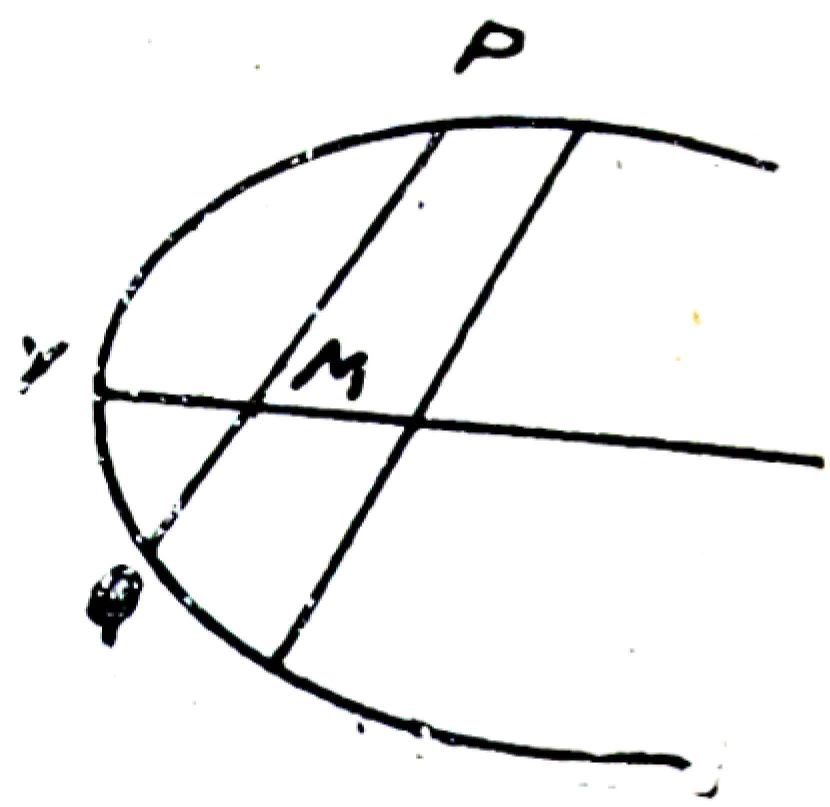
Equation (2) then becomes $y-k=\dfrac{2a}{k}(x-h)$

or $yk-2ax=k^2-2ah$......(3), the equation required.

If M be $(x_1, y_1)$, (3) becomes $yy_1 - 2ax = y_1^2 - 2ax_1$ ...(4)

**7·25. Locus of the mid-points of a system of parallel chords.**

Let PQ be a chord of the $\|$ system, $m$ the fixed slope of the $\|$ system, and $M(x, y)$, the mid-pt. of PQ.




If P be $(x_1, y_1)$, and Q, $(x_2, y_2)$

the $\left.\begin{array}{l} 2x = x_1 + x_2 \\ 2y = y_1 - y_2 \end{array}\right\}$ ...(1)

Also the equation to PQ is

$$y(y_1 + y_2) = 4ax + y_1 y_2$$

so that the slope of $PQ = \dfrac{4a}{y_1 + y_2} = \dfrac{4a}{2y} = \dfrac{2a}{y}$ from (1)

But the slope is given to be $m$.

$\therefore \quad \dfrac{2a}{y} = m$ or $\mathbf{y} = \dfrac{\mathbf{2a}}{\mathbf{m}}$ ...(2), the locus required.

Eqn. (2) is obviously a st. line $\|$ to the axis of the parabola. This line is called a **Diameter** of the parabola.

**Cor. 1. Vertex** of a diameter :

The diameter given by equation (2) intersects.

$y^2 = 4ax$ where

$$\frac{4a^2}{m^2} = 4ax \text{ or } x = \frac{a}{m^2}$$

$\therefore$ pt. of intersection of the diameter $y = \dfrac{2a}{m}$ with the parabola $y^2 = 4ax$ is $V\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ and is called the **Vertex** of the diameter.

**Cor. 2.** *Tangent at the vertex of a diameter is parallel to the system of chords bisected by it.*

The vertex V being $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$, tangent there at is

$\frac{2ay}{m}=2a\left(x+\frac{a}{m^2}\right)$ or $y=mx+\frac{a}{m}$ which is $\parallel$ to the $\parallel$ system.

**Note.** The diameter given by equation (2) meets the parabola in one pt., *viz.* $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ and appears to contradict the statement made in Art. **7·2** that every st. line meets a parabola in two points. This is actually not the case, for, from our theory of quadratic equations, we know that the equation $m^2x^2+2(mc-2a)x+c^2=0$ of that Article shall have one root infinite if the co-efficient of $x^2$, *viz.*, $m$, be zero *i.e.*, if the line be to the $x$ axis. We observe thus that a line $\parallel$ to the axis of the parabola will meet a parabola at *one pt. only at a finite distance*, the other pt. of intersection receding to an infinite distance from the origin. And a diameter being a line $\parallel$ to the axis of the parabola, the same is true of a diameter.

## Solved Examples

**Example 1.** *Find the equation of the tangent to the parabola $y^2=2x$ which is parallel to the line $3x+4y+5=0$.*

Any line parallel to $3x+4y+5=0$ is given by $3x+4y+k=0$. This shall be a tangent if

$\left(\frac{3x+k}{4}\right)^2=2x$ or $9x^2+2(3k-16)x+k^2=0$ has equal roots,

$\therefore$ $(3k-16)^2=9k^2$ whence $96k=256$ or $k=\frac{8}{3}$.

$\therefore$ $3x+4y+\frac{8}{3}=0$ or $9x+12y+8=0$ is the tangent required.

**Note.** In case the student feels like making short work of the affair, he may proceed thus :

$y=mx+a/m$ is a tangent to $y^2=4ax$ for all $m$. Here $m=-\frac{3}{4}$ and $a=\frac{1}{2}$. Substituting in $y=mx+a/m$ we get $y=-\frac{3}{4}x+\frac{1}{2}(-\frac{4}{3})$ or $y=-\frac{3}{4}x-\frac{2}{3}$ or $9x+12y+8=0$ as before.

**Example 2.** *Find the equation to the normal to the parabola $y^2=8x$ which is $\perp$ to $2x+y+1=0$.*

Any line $\perp$ to $2x+y+1=0$ has its slope equal to $\frac{1}{2}$.

$$y=mx-2am-am^3 \qquad \ldots(1)$$

is a normal to $y^2=4ax$ for all $m$.

Here $a=2$, $m=\frac{1}{2}$.

$\therefore$ (1) becomes

$$y=\tfrac{1}{2}x-4\ \tfrac{1}{2}-2\ .\ \tfrac{1}{8}$$

or $$y=\tfrac{1}{2}x-\tfrac{9}{4} \quad \text{or} \quad 2x-4y-9=0.$$

**Example 3.** *Find the equation to the common chord of $y^2=4ax$ and $x^2=4by$.*

Solving together we get $4by=\dfrac{y^4}{16a^2}$.

$\therefore$ $y(y^3-64a^2b)=0$ giving $y=0$ or $4a^{\frac{2}{3}}b^{\frac{1}{3}}$.

Then $y^2=4ax$ gives $x=0$ or $4a^{\frac{1}{3}}b^{\frac{2}{3}}$

The parabolas $\therefore$ intersect in (0, 0) and

$$(4a^{\frac{1}{3}}b^{\frac{2}{3}},\ 4a^{\frac{2}{3}}b^{\frac{1}{3}}).$$

The equation to the common chord is

$$\frac{y}{a^{\frac{2}{3}}b^{\frac{1}{3}}}=\frac{x}{a^{\frac{1}{3}}b^{\frac{2}{3}}} \quad \text{or} \quad b^{\frac{1}{3}}y=a^{\frac{1}{3}}x.$$

**Example 4.** *Find the angle at which $y^2=ax$ and $x^2=by$ intersect.*

Solving the two equations, we have

$(0, 0)$ $(a^{\frac{1}{3}}b^{\frac{2}{3}},\ a^{\frac{2}{3}}b^{\frac{1}{3}})$ as the two points of intersection.

The axis of co-ordinates are the tangents to the two parabolas respectively, at (0, 0).

$\therefore$ The curves cut orthogonally at the origin.

The slope of the tangent at $(a^{\frac{1}{3}}b^{\frac{2}{3}},\ a^{\frac{2}{3}}b^{\frac{1}{3}})$ to $y^2=ax$ is

$$\frac{x}{2a^{\frac{2}{3}}b^{\frac{1}{3}}}=\frac{a^{\frac{1}{3}}}{2a^{\frac{1}{3}}} \text{ and that of the tangent at}$$

$(a^{\frac{1}{3}}b^{\frac{2}{3}},\ a^{\frac{2}{3}}b^{\frac{1}{3}})$ to $x^2=by$ is $\dfrac{2a^{\frac{1}{3}}b^{\frac{2}{3}}}{b}=\dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

[Since the tangent, at $(x_1, y_1)$ to $x^2=by$ is $xx_1=\frac{b}{2}(y+y_1)$]

If $\theta$ be the angle of intersection,

$$\tan\theta=\frac{\dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}-\dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}}{1+\dfrac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}}=\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$$

**Example 5**. *Prove that the co-ordinates of the point of intersection of tangents to the parabola $y^2=4ax$ at $P(x_1, y_1)$ and $Q(x_2, y_2)$ are* $\left(\dfrac{y_1y_2}{4a}, \dfrac{y_1+y_2}{2}\right)$.

Hence find the locus of intersections of these tangents if (*i*) the sum of the ordinates of P and Q is constant and (*ii*) the rectangle contained by the co-ordinates of P and Q is constant.

**Sol.** Tangent at P is $yy_1=2a(x+x_1)$ ...(1)

,, ,, Q is $yy_2=2a(x+x_2)$ ...(2)

By subtraction, we have for the point of intersection,

$$y=\frac{2a(x_1-x_2)}{y_1-y_2}=\frac{\dfrac{2a}{4a}(y_1^2-y_2^2)}{y_1-y_2}$$

[$\because$ $(x_1, y_1)$ and $(x_2, y_2)$ are on the parabola.]

$$=\frac{y_1+y_2}{2} \qquad \text{...(A)}$$

Substituting for $y$ from (A) in (1), we have

$$y_1\frac{(y_1+y_2)}{2}=2a(x+x_1)=2ax+\frac{2ay_1^2}{4a}$$

$$=2ax+\frac{y_1^2}{2} \qquad \therefore \quad x=\frac{y_1y_2}{4a}. \qquad \text{...(B)}$$

The co-ordinates of the required point of intersection therefore are $\left(\dfrac{y_1y_2}{4a}, \dfrac{y_1+y_2}{2}\right)$.

(*i*) The sum of the ordinates is constant $=2K$ (say)

*i.e.* $$y_1+y_2=2K.$$

The ordinates of the point of intersection of the tangents is

$y=\frac{y_1+y_2}{2}=K \quad \therefore \quad =K$ is the locus. [By (A)]

(*ii*) If the rectangle contained by the ordinates is constant $=c^2$, $y_1y_2=c^2$. [by B]

$\therefore$ the abscissa of the point of intersection is

$$x=\frac{y_1y_2}{4a}=\frac{c^2}{4a}.$$

Hence the locus is $x=\frac{c^2}{4a}$.

**Example 6.** *Find the locus of the mid-pts. of chords of parabola $y^2=4ax$ which are normal to the curve.*

The chord whose mid-pt. is $(x_1, y_1)$ is $yy_1-2ax=y_1^2-2ax_1$. Identifying it with the normal

$$tx+y=2at+at^3$$

We get $\frac{t}{-2a}=\frac{1}{y_1}=\frac{2at+at^3}{y_1^2-2ax}$.

First two give $t=-\frac{2a}{y_1}$ and then last two give

$$y_1=\frac{y_1^2-2ax_1}{\left(\frac{-2a}{y_1}\right)2a+a\left(\frac{-8a^3}{y_1^3}\right)}$$

which gives $(y_1^2-2ax_1)y_1^2=-4a^2y_1^2-8a^4$.

$\therefore$ $(x_1, y_1)$ lies on the curve $y^2(y^2-2ax+4a^2)+8a^4=0$.

**Example 7.** *Find the locus of mid-pts. of chords of the parabola $y^2=4ax$ which subtend a right angle at the vertex and prove that these chords all pass through a fixed point on the axis of the curve.*

**Sol.** Let P, Q, the ends of a chord of the system, be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. The slopes of AP and AQ are $\frac{2}{t_1}$ and $\frac{2}{t_2}$. By the question $\frac{2}{t_1}\cdot\frac{2}{t_2}=-1$ or $t_1t_2=-4$...(*i*)

The equation to PQ is $(t_1+t_2)y=2x+2at_1t_2$ (Art. 7·22)

It meets $y=0$ where $x=-at_1t_2=4a$ by $(i)$

$\therefore$ PQ always passes through $(4a, 0)$.

Again if $(x, y)$ be the mid-pt. of PQ we have

$$2x=a(t_1^2+t_2^2) \quad \ldots(2)$$
$$2y=2a(t+t_2) \quad \ldots(3)$$

Eliminating $t_1$ and $t_2$ between (1), (2) and (3), we get $y^2=2a(x-4a)$ as the locus required.

## Exercises VII (c)

1. $(i)$ Find the points where the line $y=3x-a$ meets the parabola $y^2=4ax$.

$(ii)$ Find the points where the line $x-y=1$ meets the parabola $3y^2=4x$.

$(iii)$ Find the point of contact of $x=2y+3$ with $y^2+3x=0$. (Solve together)

$(iv)$ Prove that $x+y=1$ touches $y=x-x^2$. (Solve together)

2. Find the length of the chord intercepted by the parabola $y^2=4x$ on the line $y=x$.

3. Put down the equations of the tangents and the normals to

(a) $y^2=4ax$ at the ends of the latus rectum.

(b) to $y^2=9x$ at (4, 6).

(c) to $y^2=6x$ at the point whose ordinate is 12.

4. Find the equation to that tangent of the parabola $y^2=7x$ which is $\perp$ to $4y-x+3=0$.

[Any line $\perp$ to $4y-x+3=0$ is given by $4x+y+k=0$, solve it with $y^2=7x$, put down the condition of tangency and get $k=\frac{7}{16}$ etc......

Or Identify $4x+y+k=0$ with $y=mx+7/4m$].

5. Find the equations to $y^2=5x \parallel$ to $x+4y+1=0$. Find also the point of contact.

6. Find the point in which the normal at (2, 2) to $y^2=2x$ meets the curve again.

7. Find the equation of the tangent to (*i*) $y^2=8x$ which makes an angle of 45° with axis, (*ii*) $y^2=4ax$ which makes an angle of 60° with the axis.

8. Prove that $lx+my+n=0$ will touch $x^2=4ay$ if $al^2=mn$.

9. (*a*) Prove that $y=mx+c$ will touch $y^2=4a(x+a)$ if $c=ma+\dfrac{a}{m}$.

10. Find the point of intersection of tangents at the points whose parameters are $t_1$ and $t_2$ and prove that its ordinate is the A.M. between the ordinates of the points of contact.

11. (*a*) Show that if the tangents at $t_1$ and $t_2$ are at right angles, $t_1t_2=-1$.

(*b*) Also that the chord joining $t_1$ and $t_2$ then passes through the focus.

12. Prove that the tangents at the ends of a focal chord intersect at right angles on the directrix. [See Arts. 7·36, 7·37]

13. (*a*) Find the equation to the common tangent of $y^2=4ax$ and $x^2=4by$.

(*b*) Find the equation to the common tangent of $y^2=4ax$ and $x^2+y^2=4ax$.

14. Prove that the chord of the parabola whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$ is a normal to the curve and that its length is $6\sqrt{6}a$.

15. Prove that the area of the triangle formed by three points on a parabola is twice the area of that formed by the tangents there at.

(Take the points to be '$t_1$', '$t_2$', '$t_3$'.)

16. Prove that the ortho-centre of a triangle formed by any three tangents to a parabola lies on the directrix.

[*Hint.* Take the tangents to be $t_1y=x+at_1^2$ etc .....]

17. (*a*) Find the equation of the chord of the parabola $y^2=8x$ which is bisected at (2, −3).

(*b*) Normal at P meet the axis in T. Find the locus of the mid. pt. of PT.

(c) Normal at P meets the axis in G. Find the locus of mid. pt. of PG.

Find the locus of mid-pts. of the chords of the parabola $y^2=4ax$ which

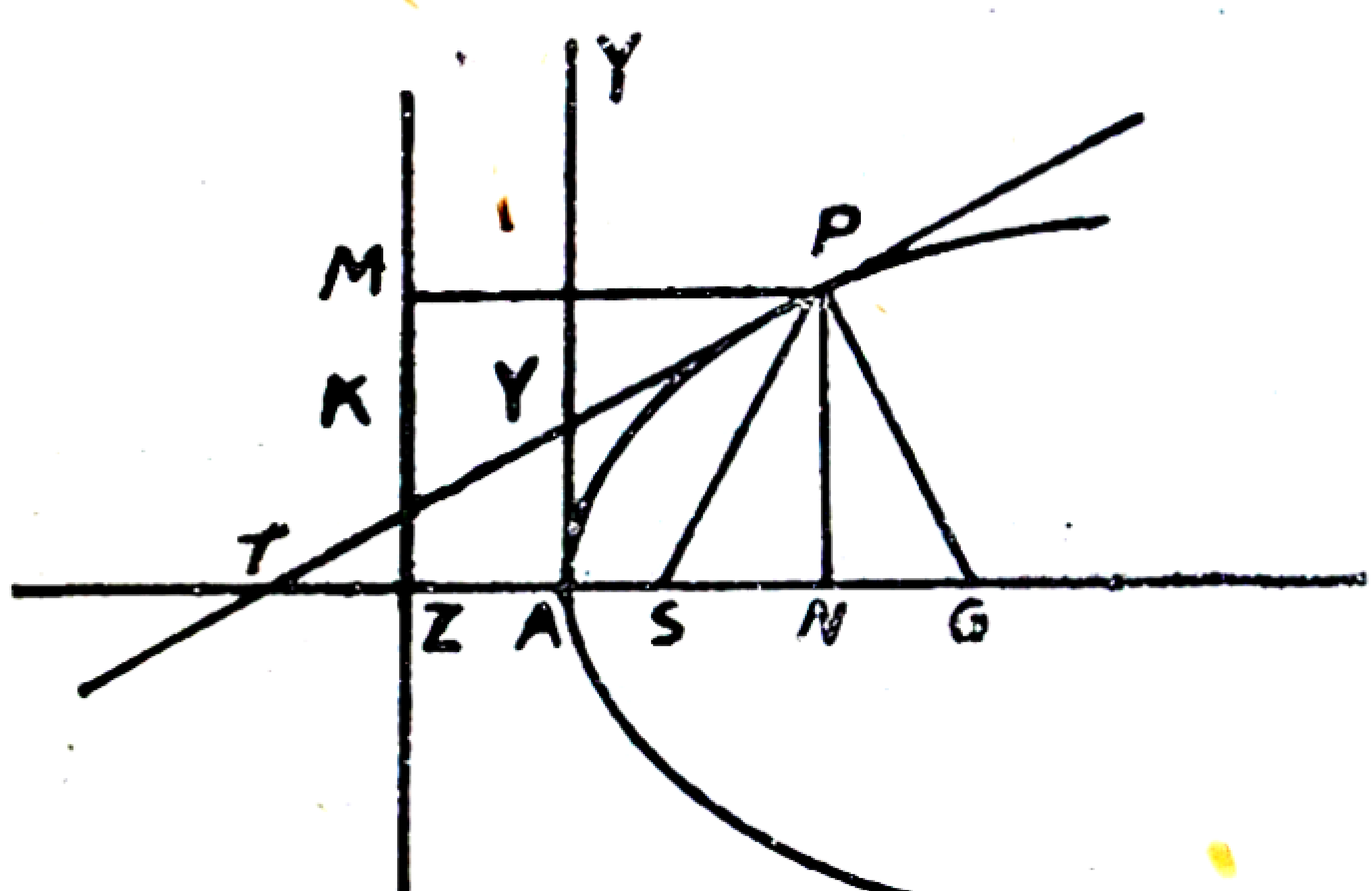
18. pass through the focus.
19. ,, ,, ,, vertex.
20. ,, ,, ,, fixed pt. $(h, k)$.

## Some Geometrical Properties of the Parabola

**Def. 1.** *The portion of the axis intercepted between a tangent and ordinate of its point of contact is called the* **sub-tangent** *for the point at which the tangent is drawn.*

**2.** *The portion of the axis intercepted between a normal and the ordinate of its foot is called the* **sub-normal** for the point at which the normal *is drawn.*

In the accompanying figure, PT is the tangent at P meeting the axis of the parabola in T and the directrix in K. PG is the normal at P meeting the axis in G. PM is perpendicular from P on the directrix, N the foot of the ordinate and Y the foot of the perpendicular from the focus S on the tangent PT. Then PT is the length of the tangent, and PG that of the normal. TN is the sub-tangent and NG the sub-normal. The equation of the parabola will throughout be taken as $y^2=4ax$.




**7·31.** *The sub-tangent for a point on the parabola is bisected at the vertex.*

Let P be $(at^2, 2at)$. Then N is $(at^2, 0)$.

Tangent at P is $ty=x+at^2$.

Where it meets $y=0$, we have $x+at^2=0$, *i.e.*, $x=-at^2$.

Hence T is $(-at^2, 0)$.

The mid-point of TN is evidently $(0, 0)$, *i.e.*, the vertex A.

**7·32.** *The sub-normal at any point of a parabola is constant and equals the semi-latus rectum.*

Normal at P$(at^2, 2at)$ is $y+tx=2at+at^3$.

Where this meets $y=0$, $x=2a+at^2$.

Hence AG$=2a+at^2$. Also AN$=at^2$.

$\therefore$ NG$=$AG$-$AN$=2a=$ semi-latus rectum.

**7·33.** *The tangent bisects the angle between the focal distance of the point and the perpendicular from the point on the directrix.*

**Sol.** We have to show that $\angle$STP$=\angle$TPM

Tangent at P$(at^2, 2at)$ is $ty=x+at^2$.

Hence if $\psi$ is the inclination of the tangent, $\tan\psi=\dfrac{1}{t}$.

If $\phi$ is the inclination of SP,

$$\tan\phi=\frac{2at}{at^2-a}=\frac{2t}{t^2-1}$$

$$=\frac{2/t}{1-1/t^2}=\frac{2\tan\psi}{1-\tan^2\psi}=\tan 2\psi$$

$\therefore\quad \phi=2\psi$

*i.e.*, $\angle$GSP$=2\angle$STP. But $\angle$GSP$=\angle$STP$+\angle$SPT also.

$\therefore\quad \angle$SPT$=\angle$STP$=\angle$TPM [$\because$ PM $\parallel$ ST].

**7·34.** *The portion of the tangent intercepted between the directrix and the point of contact subtends a right angle at the focus.*

[We have to show that $\widehat{\text{KSP}}=1$ rt. $\angle$]

Tangent at P is $ty=x+at^2$.

Where this meets the directrix $x=-a$,

$$ty=-a+at^2 \quad\text{or}\quad y=\frac{a}{t}(t^2-1)$$

Thus K is $\left\{-a, +\dfrac{a}{t}(t^2-1)\right\}$

$\therefore$ Slope of $KS = \dfrac{0 - a/t(t^2-1)}{a+a} = -\dfrac{1}{2t}(t^2-1)$.

Slope of $SP = \dfrac{2at - 0}{at^2 - a} = \dfrac{2t}{t^2-1}$.

$\therefore$ the product of the two slopes $= -1$, the two lines are at right angles *i.e.*, $K\hat{S}P = 1$ rt. $\angle$.

**7·35.** *The locus of the foot of the perpendicular from the focus on any tangent to a parabola is the tangent at the vertex.*

[We have to show that the abscissa of the foot of the perpendicular is zero ; for it is only then that it will lie on the tangent at the vertex which is the $y$-axis.]

Equation of the tangent at P is

$$ty = x + at^2 \quad ...(1)$$

Equations of the line through S$(a, 0)$ perpendicular to it is

$$y = -t(x-a) \quad \text{or} \quad y = -tx + at \quad ...(2)$$

Multiplying (2) by $t$ and subtracting from (1),

$$0 = (1+t^2)x \quad \text{or} \quad x = 0.$$

*i.e.*, the abscissa of the foot of the perpendicular from S on the tangent is zero.

Hence the required locus is $x = 0$ which is the tangent at the vertex.

**7·36.** *Tangents at the ends of a focal chord are at right angles.*

Let '$t_1$' + '$t_2$' be the ends of a focal chord. Equation of the chord is $(t_1+t_2)y = 2x + 2at_1t_2$.

The focus $(a, 0)$ lies on it.

$$\therefore \quad 0 = 2a + 2at_1t.$$

$$\text{or} \quad t_1t_2 = -1. \quad ...(1)$$

The tangents at the extremities '$t_1$' '$t_2$' are

$$\left.\begin{array}{l} t_1y = x + at_1^2 \\ t_2y = x + at_2^2 \end{array}\right\} \quad ...(2)$$

Product of the slopes of the two tangents$=\frac{1}{t_1}, \frac{1}{t_2}=-1$ from (i).

Hence the two tangents are at right angles.

**7·37.** *Tangents at the ends of a focal chord intersect on the directrix.*

Multiplying the equations (2) above by $t_2$ and $t_1$ respectively and then subtracting we get for the abscissa of the point of intersection of the tangents

$$0=(t_2-t_1)x+at_1t_2(t_1-t_2).$$

or $\quad x-at_1t_2=0 \qquad [\because t_1\neq t_2].$

or $\quad x=at_1t_2=-a$ from (1) Art. 7·36.

Thus the tangents at the ends of a focal chord intersect on the directrix.

Combining the results of 7·36 and 7·37 we can state that *tangents at the ends of a focal chord intersect at right angles on the directrix.* [Cf. Q 12 VII C].

**7·38.** *The semi latus rectum is a harmonic mean between the segments of a focal chord.*

$$\frac{1}{SP}+\frac{1}{SQ}=\frac{2}{SL}.$$

If '$t_1$' '$t_2$' are the extremities P and Q of a focal chord,

$$t_1t_2=-1 \text{ or } t_2=-\frac{1}{t_1}.$$

Hence P is $(at_1^2, 2at_1)$ and Q is $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$

$SP=a(t_1^2+1)$ ; $\qquad [\because SP=a+x=a+at_1^2.$

$SQ=a(t_2^2+1)=a\left(\frac{1}{t_1^2}+1\right)=\frac{a}{t_1^2}(1+t_1^2)$

$$\therefore \frac{1}{SP}+\frac{1}{SQ}=\frac{1}{a(t_1^2+1)}+\frac{t_1^2}{a(t_1^2+1)}=\frac{1+t_1^2}{a(t_1^2+1)}=\frac{1}{a}$$

$$=\frac{2}{2a}=\frac{2}{SL}.$$

**7·39.** *The perpendicular from the focus on any tangent is a mean proportional between the focal distances of the vertex and the point of contact.*

*i.e.*, $SY^2 = AS.SP$.

Tangent at P is $ty = x + at^2$.

Also $SP = a(t^2+1)$.

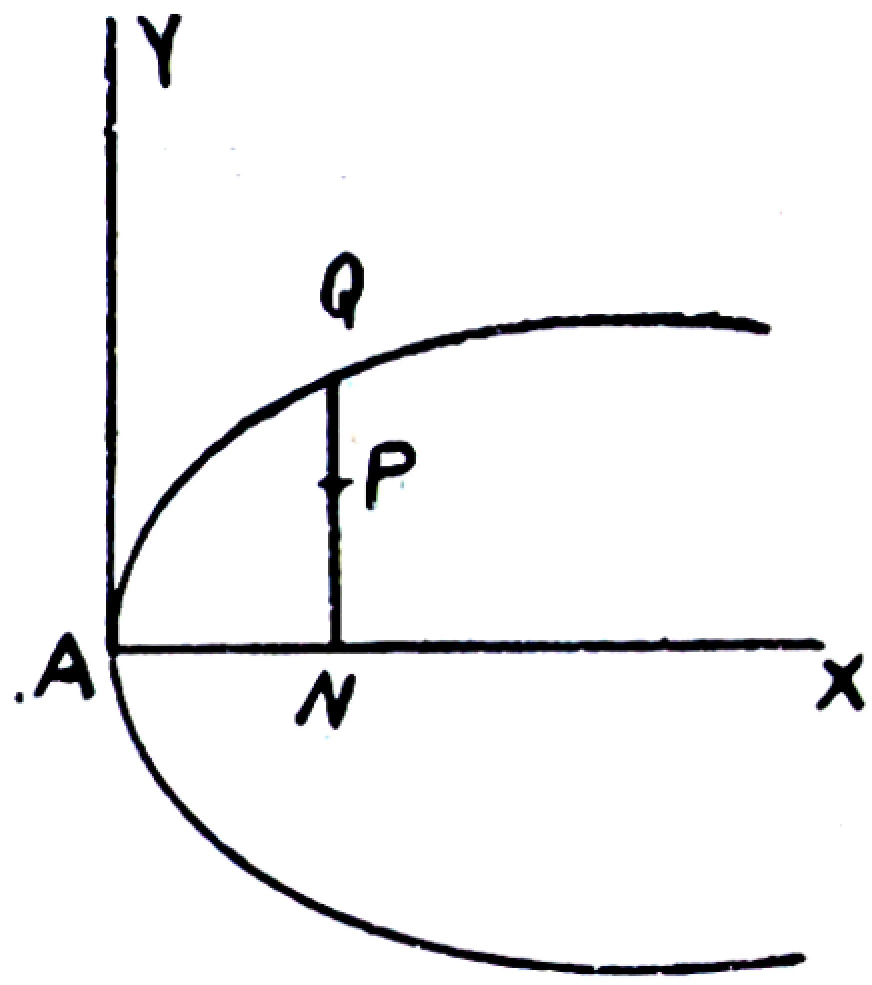
Perpendicular from $S(a, 0)$ on the tangent

$$SY = \frac{a+at^2}{\sqrt{1+t^2}} = a\sqrt{1+t_2}.$$

$$\therefore \quad SY^2 = a^2(1+t^2) = a.a(1+t^2)$$
$$= AS.SP.$$

## 7·4. Position of a point relative to a parabola.

Let there be a point $P(x_1, y_1)$.




Draw PN $\perp$ from P on the axis and let it meet (produced if necessary) the parabola in Q If Q be $(x_1, y')$, then since $Q(x_1, y')$ lies on the parabola $y'^2 - 4ax_1 = 0$. ...(1)

Now P will be outside the parabola if NP>NQ or $y_1 > y'$ or $y_1 > y'^2$ or if $y_1^2 - 4ax_1 > y'^2 - 4ax_1 = 0$ *i.e.*, if $y_1^2 - 4ax_1 > 0$. So also P will be inside the parabola if $y_1^2 - 4ax_1 < 0$. When P lies on the parabola $y_1^2 - 4ax_1 = 0$.

## 7·41. Tangents from a point.

*Through a pt. two tangents can be drawn to a parabola and they will be real and distinct, coincident, or imaginary according as the pt. is outside, on or inside the parabola.*

We know that $y = mx + \frac{a}{m}$ ...(1)

touches the parabola $y^2 = 4ax$ whatever $m$ may be.

It will pass through a particular point

$$P(x_1, y_1) \text{ if } y_1 = mx_1 + \frac{a}{m}.$$

**The equation which may be written as**

$$x_1m^2-y_1m+a=0 \qquad \ldots(2)$$

gives us the slopes of the tangent that pass through $P(x_1, y_1)$. Since this is a quadratic in $m$, it will give us two values of $m$ which when substituted in (1) will give us the tangents that pass through $P(x_1, y_1)$. Further, the tangent will be real, coincident or imaginary according as (2) has real, equal, or imaginary roots.

*i e.*, according as $y_1^2-2ax_1>, =$ or $<0$

*i.e.*, according as $(x_1, y_1)$ is outside, on or inside the parabola.

**Example**. *Find the equations of the two tangents to the parabola $y^2=2x$ drawn from the point $(-4, 1)$.*

Here $a=\frac{1}{2}$

$$\therefore \quad y=mx+\tfrac{1}{2}m \qquad \ldots(1)$$

will be a tangent for all values of $m$.

It will pass through the pt. $(-4, 1)$ if

$$1=-4m+\tfrac{1}{2}m.$$

or $$8m^2+2m-1=0$$

whence $$m=-\tfrac{1}{2}, \tfrac{1}{4}.$$

Substituting for $m$ in (1), the two tangents are

$$x+2y+2=0$$

and $$x-4y+8=0$$

**7·42.** *Locus of the point of intersection of tangents to a parabola inclined at a constant $\angle a$.*

Let $y^2=4ax$ be the parabola and $(x_1, y_1)$, the point of intersection of the tangents.

$$y=mx+\frac{a}{m} \qquad \ldots(1)$$

touches it for all values of $m$,

(1) will pass through $(x_1, y_1)$ if

$$y_1=mx_1+\frac{a}{m} \qquad \ldots(2)$$

Eqn. (2) which may be rewritten as

$$x_1m^2-y_1m+a=0$$

gives the slopes of the two tangents to the parabola that pass through the point $(x_1, y_1)$.

If $m_1, m_2$ be the roots of (3),

$$m_1+m_2=\frac{y_1}{x_1} \qquad \ldots(4)$$

$$m_1m_2=\frac{a}{x_1} \qquad \ldots(5)$$

By the equation;

$$\tan a=\frac{m_1-m_2}{1+m_1m_1}$$

$$=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

$$=\frac{\sqrt{\frac{y_1^2}{x_1^2}-\frac{4a}{x_1}}}{1+\frac{a}{x_1}}$$

$$=\frac{\sqrt{y_1^2-4ax_1}}{x_1+a}$$

which gives $(y_1^2-4ax_1)\cot^2 a=(x_1+a)^2$

$\therefore$ $(x_1, y_1)$ lies on

$$(y^2-4ax)\cot^2 a=(x+a)^2 \qquad \ldots(6)$$

the locus required.

**Cor.** When $\alpha=90°$, *i.e.*, the tangents are perpendicular to each other, $m_1m^2=-1$

and then (5) gives,

$$\frac{a}{x_1}=-1$$

so that $x+a=0$, the locus required.

It is no other than the directrix of the parabola.

**Notes.** It could also be got from (6) by making $\alpha=90°$.

2. Many other cases of locus of intersection of tangents satisfying other conditions will be easily disposed of by (4) and (5) combined with the conditions given in any particular case. (See Questions 3—7) below.

3. We shall require the following few facts from Algebra in the discussion of the normals from a point to a parabola.

Just as a quadratic like $ax^2+bx+c=0$ has two roots and its L. H. S. *viz.* $ax^2+bx+c\equiv a(x-\alpha)(x-\beta)$ identically where $\alpha$, $\beta$ are its roots, a cubic such as $ax^2+bx^2+cx+d=0$ has three roots and its L. H. S. *viz.*

$$ax^3+ax^2+cx+d\equiv a(x-\alpha)(x-\beta)(x-\gamma) \qquad \ldots(1)$$

identically where $\alpha$, $\beta$, $\gamma$, are the roots of the cubic.

(1) may be written as,

$$x^2-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$$

$$\equiv x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a} \qquad \ldots(2)$$

Comparing co-efficients of like powers of $x$ on both sides of this identity, we get

$$\alpha+\beta+\gamma=-\frac{b}{a},\quad \alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a},\quad \alpha\beta\gamma=-\frac{d}{a}.$$

### 7·43. Normals through a point.

Let $y^2=4ax$ be the parabola and $(x_1, y_1)$ the point,

$$y=mx-2am-am^3 \qquad \ldots(1)$$

is a normal to the parabola for all values of $m$.

It will pass through the point $(x_1, y_1)$ if

$$y_1=mx_1-2am-am^3 \qquad \ldots(2)$$

Eqn. (2) which may be rewritten as

$$am^3+(2a-x_1)m+y_1=0 \qquad \ldots(3)$$

gives the slopes of the normals that pass through the point $(x_1, y_1)$. Since (3) is a cubic in $m$, it will give us three values of $m$, which when substituted in (1) will give us the three normals that can be drawn from the point, $(x_1, y_1)$ to the parabola.

**Note.** We know that imaginary roots occur in conjugate pairs, $\therefore$ of the three roots of (2), at least one will be real so

that, of the three normals through a point, at least one will be real.

**Example.** *Find the locus of the point of intersection of normals to the parabola $y^2=4ax$, two of which make complementary angles with the axis.*

If $m_1$, $m_2$, $m_3$ be the roots of equation (3) above,

$$m_1+m_2+m_3=0 \qquad \ldots(1)$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{2a-x_1}{a} \qquad \ldots(2)$$

$$m_1m_2m_3=-\frac{y_1}{a} \qquad \ldots(3)$$

By the question, $m_1m_2=1$

$\therefore$ (3) gives $m_3=-\dfrac{y_1}{a}$

But $m_3$ is a root of $am^3+(2a-x_1)m+y_1=0$ ...(4)

$$\therefore\ a\left(\frac{-y_1}{a}\right)^3+(2a-x_1)\left(\frac{-y_1}{a}\right)+y_1=0$$

or $$-\frac{{y_1}^3}{a^2}+\frac{x_1y_1}{a}+y_1+-2y_1=0.$$

or $${y_1}^3+ax_1y_1-a^2y_1=0 \quad \text{or} \quad {y_1}^2=a(x_1-a) \qquad \ldots(5)$$

(5) being a relation between the co-ordinates $x_1$, $y_1$ of the point from which the normals are being drawn, the equation to the locus is $y^2=a(x-a)$.

It is a parabola whose latus rectum is $a$ and vertex at $(a, 0)$.

**Note.** The locus of the point of intersection of the normals, satisfying other conditions, may similarly be obtained from equations (1), (2), (3) combined with the conditions obtaining in a particular question. It is a question of elimination of $m_1$, $m_2$, $m_3$.

**Cor. 1.** Equation (2) above, incidentally proves that the slopes of the three normals from a point to a parabola, add upto zero.

**Cor. 2.** Further if $y_1$, $y_2$, $y_3$, be the ordinates of the feet of the three normals from a point to a parabola, **Cor. 1.** Art. 7·22, (*c*) gives

$$y_1+y_2+y_3=-2a(m_1+m_2+m_3)=0 \text{ by (1).}$$

Hence the sum of the ordinates of the feet of the three normals from a point is also zero.

### Exercises VII (d)

1\. (*a*) Find the equations of the two tangents to the parabola $y^2=4x$ from the point (3, −4).

(*b*) Find the tangents to the parabola $2y^2-3x=0$ drawn from $(-12, \frac{3}{2})$.

2\. Prove that if $\theta$ be the angle between the tangents that can be drawn from $(x_1, y_1)$ to $y^2=4ax$,

then $\tan\theta=\dfrac{\sqrt{y_1^2-4ax_1}}{x_1+a}$

Find the locus of the point of intersection of tangents to the parabola $y^2=4ax$ drawn from an external point P when these tangents make $\angle$s $\theta_1$, $\theta_2$ with the axis and,

3\. (*a*) $\tan\theta_1+\tan\theta_2=b$.

(*b*) $\tan\theta_1, \tan\theta_2=c$.

4\. $\cot\theta_1+\cot\theta_2=d$.

5\. $\theta_1+\theta_2=2a$.

6\. $\tan^2\theta_1+\tan^2\theta_2=g$.

7\. Find the locus of intersection of tangents to a parabola meeting (*i*) at an $\angle 45°$, (*ii*) at an $\angle 60°$.

[Particular cases of 7·42]

8\. (*a*) Find the equation to the normals to $y^2=4ax$ drawn from the point $(0, 3a)$.

[$y=mx-2am-am^3$ is a normal. The rest as in solved example Art. 7·41].

(*b*) Find the equations of the normals from $(5a, 2a)$ to $y^2=4ax$.

Find the locus of a pt. P when the three normals from it to a parabola are such that

9\. Two of them are at rt. angles.

10. Two of them are such that the product of their slopes equals 2.

(A beautiful statement of Q. 10. *Prove that the locus of a point, two of the three normals from which have the product of their slopes equal to 2, is the parabola itself*).

11. The sum of the three angles made by them with the axis is constant (say $k$).

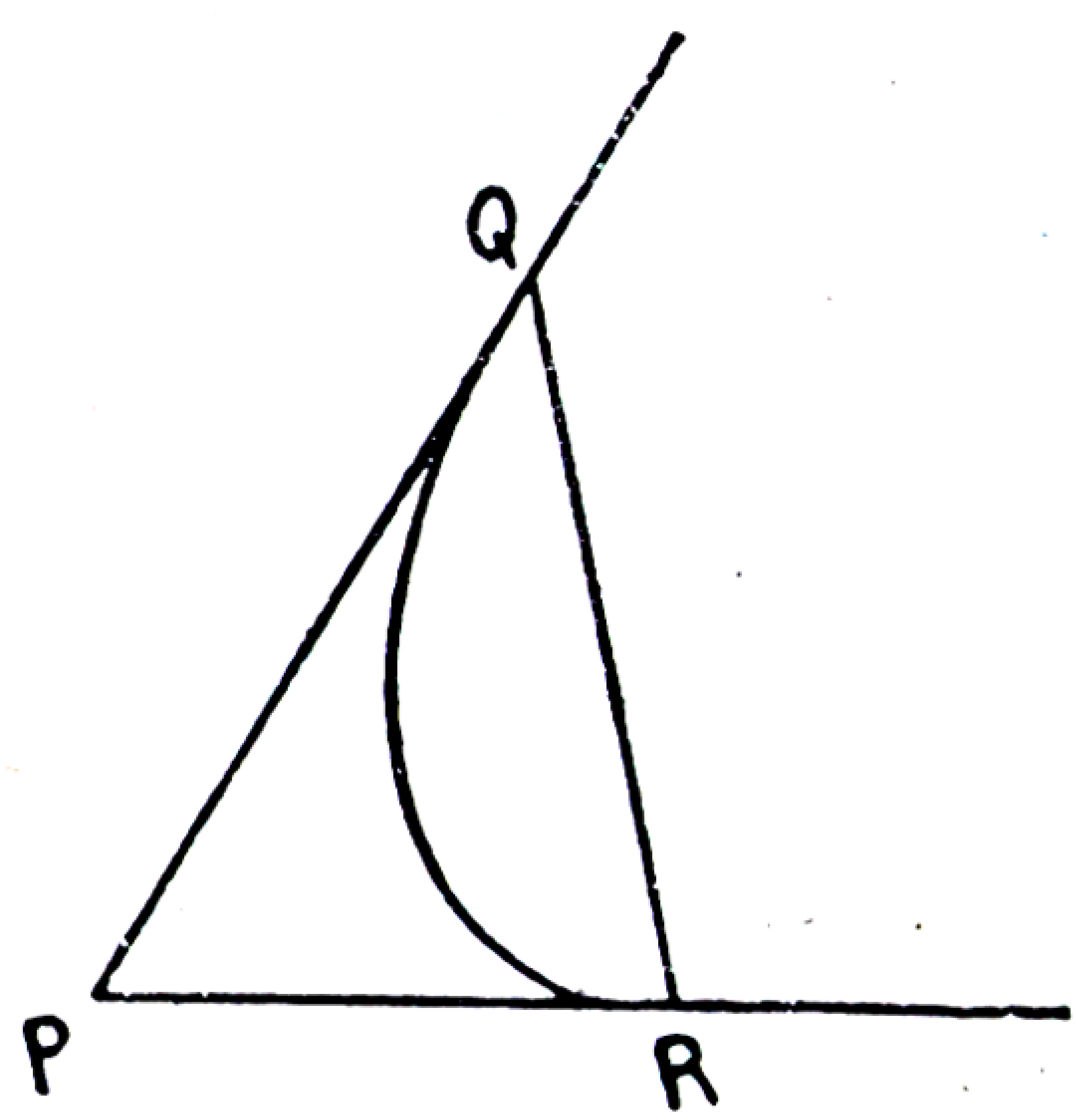
12. Prove that the centroid of the triangle formed by the feet of the normals from a point to a parabola lies on the axis of the parabola. (Use **Cor. 2.**)

## 7·5. Chord of Contact.

**Def.** *If Q and R be the points of contact of tangent to a parabola from a point P outside it, QR is called the chord of contact of P w. r. to the parabola.*

### Equation of the Chord of Contact

Let there be a point $P(x_1, y_1)$. Let PQ, be tangent to the parabola from P.




Let Q be $(x_2, y_2)$ and R $(x_3, y_3)$

Then QP and RP are given by

$$yy_2=2a(x+x_2) \qquad \ldots(2)$$

and

$$yy_3=2a(x+x_3) \qquad \ldots(3)$$

(2) and (3) both pass through $P(x_1, y_1)$

$$\therefore \quad y_1y_2=2a(x_2+x_1) \qquad \ldots(4)$$

and

$$y_1y_3=2a(x_3+x_1) \qquad \ldots(5)$$

From (4) and (5) it is clear that the points Q $(x_2, y_2)$ and R $(x_3, y_3)$ lie on the line $yy_1$

$$=2a(x+x_1) \qquad \ldots(6)$$

which therefore is the chord of contact of P.

**Note.** Equation (6) is of the same form as the equation of the tangent at P $(x_1\, y_1)$ if it comes to lie on the parabola. This suggests that a tangent may be a particular case of the chord of contact. That this is actually so, may be seen from the fact that the tangents from P (when P is on the parabola) coincide with the tangent at P so that their points of contact

coincide at P ; and the chord of contact of P is nothing but the tangent at P.

When P is inside the parabola, the tangents from P are imaginary (Art. 7·41) : so also shall be the points of contact. But the form of equation (6) shows that if $(x_1, y_1)$ be real, (6) gives a real line whether $(x, y_1)$ lie outside, or inside the parabola. So that we could look upon (6) as the chord of contact of P $(x_1, y_1)$ even when P is inside the parabola. Only now it shall pass through the imaginary points of contact of the two imaginary tangents from P.

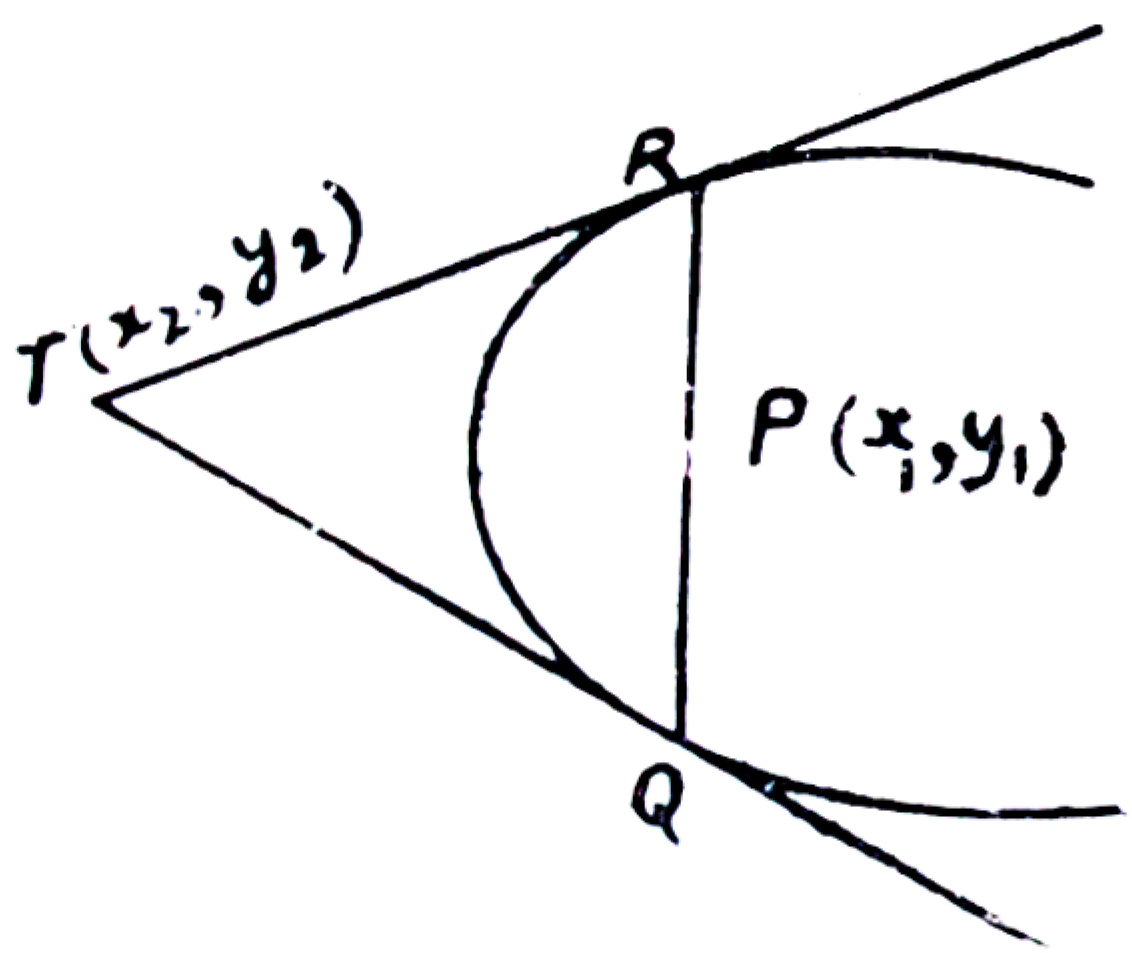
### 7·52. Pole and polar.

**Def.** *The* **polar** *of a point w. r to a parabola is the locus of the point of intersection of tangents to the parabola at the ends of any chord drawn through the given point.*

*The point is called* **pole** *of its polar.*

### 7·53. The equation of the Polar of a point.

Let P be $(x_1, y_1)$ and the parabola be $y^2=4ax$ .. (1)




also let QR be any chord through P and let the tangents at Q and R meet in T. Then the locus of T is the polar of P.

If T be $(x_2, y_2)$ then QR which is the chord of contact of T is given by

$$yy_2=2a(x+x_2) \quad \text{... (2)}$$

(2) goes through $P(x_1, y_1)$

$$\therefore y_1y_2=2a(x_2+x_1) \quad \text{... (3)}$$

(3) is a relation between the co-ordinates $x_2$, $y_2$ of the point T.

$\therefore$ the equation to the locus of T is

$$yy_1=2a(x+x_1) \quad \text{... (4)}$$

the polar required.

*Note.*—Equation (4) is the same as (6) of the last article. This suggests that the polar of a point *w. r.* to a parabola coincides with the chord of contact when the point is outside and with the tangent when the point is on the parabola.

(Why ?

**7·54.** *If the polar of a point P w. r. t. a parabola goes through Q, then will the polar of Q pass through P.*

Let the parabola be $y^2=4ax$ and P be $(x_1, y_1)$ and Q $(x_2, y_2)$. Then $\because$ the polar of P $(x_1, y_1)$ *viz.*, $yy_1=2a(x+x_1)$ passes through Q$(x_2, y_2)$ we have

$$y_1y_2=2a(x_2+x_1)$$

which is exactly the condition for the polar of Q $(x_2, y_2)$ *viz.*, $yy_2=2a(x+x_2)$ to pass through P $(x_1, y_1)$. Hence the result.

**Def.** Two such points are called *conjugate points w. r. t.* the parabola.

**7·55** *To find the pole of the line* $lx+my+n=0$ ... (1)

*w. r. t. the parabola* $y^2=4ax$ ... (2)

If $(x_1, y_1)$ be the pole of (1) *w. r. t.* to (2), (1) must be the same as $yy_1=2a\ (x+x_1)$.

Comparing co-efficients $\dfrac{2a}{l}=-\dfrac{y_1}{m}=\dfrac{2ax_1}{n}$

which gives $x_1=\dfrac{n}{l}$ and $y_1=-\dfrac{2am}{l}$

**7 56.** *If the pole of a given line $L_1$ is given to lie on another line $L_2$ then will the pole of $L_2$ lie on $L_1$.*

If equations to $L_1$ and $L_2$ be

$$l_1x+m_1y+n_1=0 \quad \dots (1)$$

$$l_2x-m_2y+n_2=0 \quad \dots (2)$$

the pole of (1) *w. r. t.* the parabola is $\left(\dfrac{n_1}{l_1}, \dfrac{-2am_1}{l_1}\right)$

(by Art. 7·55)

It lies on (2) $\therefore\ l_2n_1-2am_1m_2+l_1n_2=0$ ... (3)

which is also the condition for the pole

$$\left(\frac{n_2}{l_2}, \frac{-2am_2}{l_2}\right) \text{ of (2) to lie on (1)}$$

**Def.** Two such lines are called conjugate lines *w. r. t.* the parabola.

## Exercises VII (e)

**1.** Find the polar of (2, 1) *w. r. t.* $y^2=6x$.

**2.** Prove that the directrix of a parabola is the polar of focus.

**3.** (*i*) Find the pole of $3x+4y-5=0$ *w. r. t.* $y^2=4x$.
(*ii*) Find the pole of $x-2y+3a+0$ *w. r. t.* $y^2=4ax$.

**4.** If the polars of P and Q *w. r. t.* a parabola meet in R, then R is the pole of the line PQ.

**5.** Prove that the chord of contact of $\perp$ tangents to a parabola passes through the focus [See Q 11 Exercises VI c].

**6.** Find the points of contact of tangents from (2, 3) to $y^2=4x$ and prove that the corresponding normals intersect on the parabola.

**7.** Find the condition (1) for two points to be conjugate *w. r. t.* a parabola (2) for two lines to be conjugate *w. r. t.* a parabola. [See equation (2) of Art. 7·54. and equation (3) of Art. 7·56]

We shall conclude this chapter with a few solved examples.

**Example 1.** Find the locus of the poles of tangents to $y^2=4ax$ *w. r. t.* $y^2=4bx$.

$ty=x+at^2$......(1) is a tangent to $y^2=4at$ for all $t$

If $(x_1, y_1)$ be the pole of (1) *w. r. t.* $y^2=4bx$, then (1) must be the same as $yy_1=2b(x+x_1)$.

Comparing co efficients we get $$\frac{y_1}{t}=\frac{2b}{1}=\frac{2bx_1}{at^2} \qquad \dots (2)$$

Variation of $t$ gives different tangents to $y^2=4ax$ and they in turn then have different poles *w. r. t.* $y^2=4bx$.

To find the locus of $(x_1, y_1)$ then, we must eliminate $t$ between equations (2).

From the first two, $t=\dfrac{y_1}{2b}$ and from the last two $t^2=\dfrac{x}{a}$.

$$\therefore \quad \frac{y_1^{\,2}}{4b^2}=\frac{x_1}{a}$$

so that $(x_1, y_1)$ lies on $y^2=\dfrac{4b^2}{a}x$.

**Example 2.** Prove that the line $Ax+By+C=0$ will be a normal to the parabola $y^2=4ax$ if $aA^3+2aAB^2+B^2C=0$

$$tx+y=2at+at^3 \qquad \dots (1)$$

Eqn. (1) is a normal to the parabola for all values of $t$. $Ax+By+C=0$ will be a normal if it is the same as (1).

Comparing co-efficients we get $\frac{t}{A}=\frac{1}{B}=\frac{2at+at^3}{-C}$

$$\therefore \quad t=\frac{A}{B} \text{ and } \frac{-C}{B}=2at+at^3.$$

so that $\frac{-C}{B}=2a\frac{A}{B}+\frac{aA^3}{B^3}$

or $aA^3+2aAB^2+B^2C=0$.

**Example.** Prove that the locus of poles of normal chords of the parabola $y^2=4ax$ is the curve.

$y^2(x+2a)+4a^3=0$.

$$tx+y=2at+at^3 \qquad \dots (1)$$

(1) is a normal for all values of $t$.

If $(x_1, y_1)$ be the pole of (1) *w. r. t.* $y^2=4ax$.

(1) must be the same as $yy_1=2ax+2ax_1$. Comparing co-efficients we get

$$\frac{t}{-2a}=\frac{1}{y}=\frac{2at+at^3}{2ax_1}=\frac{2t+t^3}{2x_1} \qquad \dots (2)$$

As $t$ varies, the normal changes its position and so does its pole. To get the locus of the pole, therefore we must eliminate T between equations (2) and (3). We have

$$\frac{2x_1}{y_1}=2t+t^3$$

and $\qquad t=-\frac{2a}{y_1}$

so that $\frac{2x_1}{y_1}=-\frac{4a}{y_1}-\frac{8a^3}{y_1^3}$

or $x_1y_1^2+2ay_1^2+4a^3=0$

$\therefore \quad (x_1, y_1)$ describes the locus whose equation is

$$y^2(x+2a)+4a^3=0.$$

## Exercises VII (f)

**1.** Prove that the equation $y^2+2Ax+2By+c=0$ represents a parabola whose axis is parallel to the axis of $x$. Find its vertex and the equation to its latus rectum.

**2.** If on a given base triangles be described such that the sum of the tangents of the base angles is constant, prove that the locus of the vertices is a parabola.

**3.** Prove that the locus of the centre of a circle which intercepts a chord of given length $2a$ on the axis of $x$ and passes through a given point on the axis of $y$ distant '$b$' from the origin, is the parabola $x^3-2yb+b^2=a^2$.

**4.** Find the point of intersection of the normals at the points '$t_1$' and '$t_2$'.

**5.** If O be any point on the axis of the parabola $y^2=4ax$ and POQ be any chord passing through O and PM and QN be the ordinates of P and Q ; prove that

(*i*) $AM.AN=AO^2$ (*ii*) $PM.QN=-4a.AO$.

**6.** Prove that the length of the chord of the parabola $y^2=4ax$ which is a normal at the point $(a, 2a)$ is $8a\sqrt{2}$.

**7.** (*a*) The normal at the point '$t$' meets the parabola again in the point '$t_1$' ; prove that $t_1=-t-\dfrac{2}{t}$.

(*b*) Prove that the normal chord at the point $t$ whose ordinate is equal to its abscissa subtends a rt. $\angle$ at the focus.

[**Hint.** $t=2$, $t_1=-3$, from (*a*)].

**8.** Prove that the semi-latus rectum is a harmonic mean between the segments of any focal chord.

[We have to prove that $\dfrac{1}{SP}+\dfrac{1}{SQ}=\dfrac{2}{SL}$]

**9.** The normal at any point P meets the axis in G and the tangent at the vertex in G′ ; if A be the vertex and the rectangle AGQG′ be completed, prove that the locus of Q is $x^3=2ax^2+ay^2$.

**10.** Through the vertex of the parabola $y^2=4ax$, two chords AP and AQ are drawn, and the circles on AP and AQ as diameters intersect in R. Prove that, if $\theta_1$, $\theta_2$ and $\phi$ be the angles made with the axis by the tangents at P and Q and by AR, then $\cot\theta_1+\cot\theta_2+2\tan\phi=0$.

**11.** Prove that the poles w. r. t. $y^2=-4ax$ of tangents to $y^2=4ax$ lie on the latter parabola.

**12.** Prove that the locus of poles of tangents to $y^2=4ax$ w.r.t. the circle $x^2+y^2=2ax$ is the circle $x^2+y^2=ax$.

# CHAPTER VIII

## THE ELLIPSE

**8·1. Def.** *An ellipse is the locus of a point which moves so that its distance from a fixed point (called the focus) bears a constant ratio (which is less than unity and is known as the eccentricity) to its distance from a fixed line (called the directrix.)*

**8·11.** *To find the equation of an ellipse with any point as focus and any line as directrix.*

Let S $(a, b)$ be the focus, $lx+my+n=0$, the directrix and $e$, the eccentricity. Let $(x, y)$ be any pt. on the ellipse and PM, the perpendicular from P on the directrix. Then by definition,

$$\frac{SP}{PM}=e \text{ or } SP^2=e^2\,PM^2 \ldots (1)$$

Now $SP^2=(x-a)^2+(y-b)^2$

and $PM^2=\dfrac{(lx+my+n)^2}{l^2+m^2}$ $(l^2+m^2)$

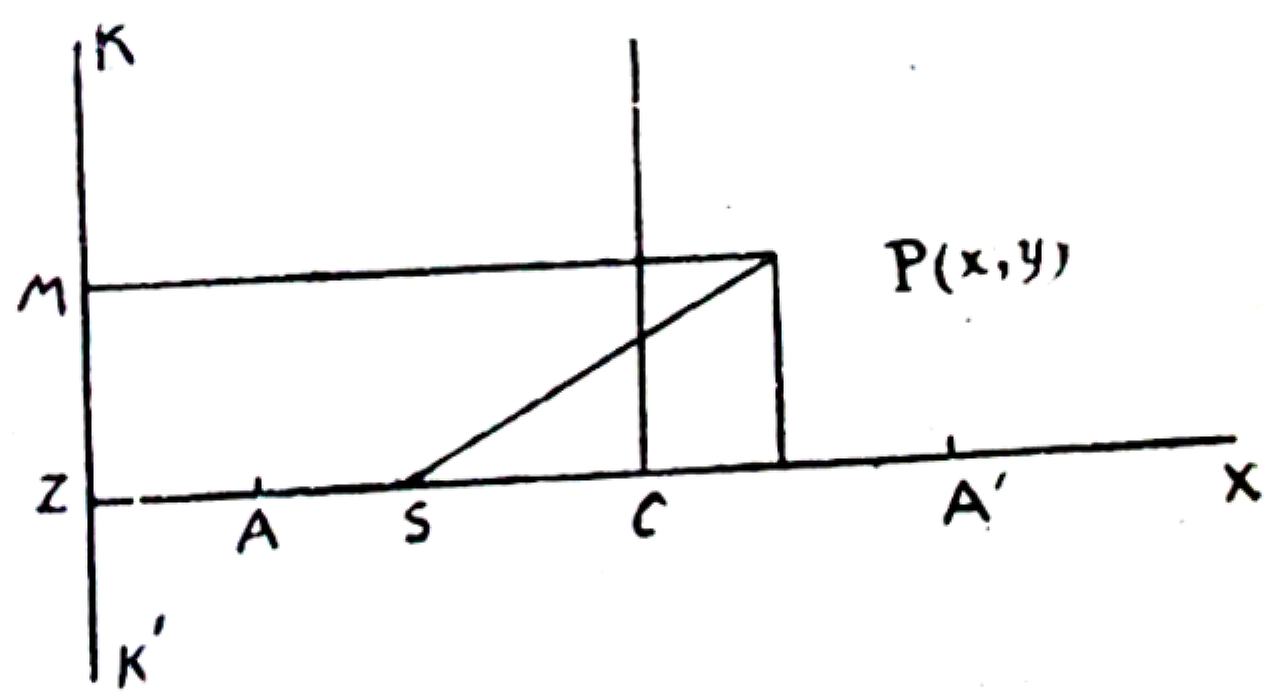
$\therefore$ (1) becomes $[(x-a)^2+(y-b)^2]=e^2(lx+mx+n)^2$ which is the equation required.

This again being as cumbersome as that of the parabola in Art. 7·11 we proceed to simplify it with a proper choice of the axes.

**8·12. Equation to an ellipse in a simplified form.**

Let S be the focus, MZ, the directrix and $e$, the eccentricity. Let A A′ divide SZ internally and externally in the ratio $e : 1$. Then since

$$\frac{SA}{AZ}=e=\frac{SA'}{A'Z'}$$




A, A′ are pts. on the ellipse. Also since $e<1$, A must lie

on the right of A. Bisect AA′ at C and let AA′$=2a$ so that AC=CA′$=a$. Take C as the origin, AA′, as the $x$-axis and a line through C $\perp$ AA′, as the $y$-axis.

Let P $(x, y)$ be any pt. on the ellipse so that CN$=x$, NP$=y$. Through P draw PM $\perp$ to the directrix.

We require the length CS and CZ in our work and proceed to calculate them. By construction,

$$SA=e,\ AZ\ i.e.,\ (CA-CS)=e\ (CZ-CA) \qquad \ldots\ldots(1)$$

$$SA'=e.\ A'Z\ i.e.,\ (CA+CS)=e\ (CZ+CS) \qquad \ldots\ldots(2)$$

$$[\because\ CA=CA']$$

Adding (1) and (2), $2CA=2e.\ CZ$ or $CZ=\dfrac{CA}{c}=\dfrac{a}{e}$ ......(3)

$\therefore$ Equation of the directrix is $x=-\dfrac{a}{e}$ or $x+\dfrac{a}{e}=0$.

Subtracting (1) from (2) we get

$$2CS=2e.\ CA \text{ or } CS=eCA=ea.$$

$\therefore$ S is $(-ae, 0)$.

From the definition of the ellipse

$$SP=ePM \quad\text{or}\quad SP^2=e^2PM^2 \qquad \ldots\ldots(4)$$

$$SP^2=(x+ae)^2+y^2 \text{ and } PM^2=\left(x+\frac{a}{e}\right)^2$$

$\therefore$ Eqn. (4) gives $(x+ae)^2+y^2=e^2\left(x+\dfrac{a}{e}\right)^2$

$$=e^2x^2+2aex+a^2$$

or $\quad x^2+2aex+a^2e^2+y^2 \quad =e^2x^2+2aex+a^2$

or $\quad (1-e^2)x^2+y^2 \quad =a^2(1-e^2)$

or $$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e)^2}=1 \qquad \ldots\ldots(5)$$

Since $e<1$, $a^2(1-e^2)$ is positive, say $b^2$.

With this substitution, (5) may be written as

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1} \qquad \ldots\ldots(6)$$

which is the equation required.

**Note 1.** If AA′ were the $y$-axis, S would be $(0, -ae)$ and $PM = ZN = ZC + CN = \frac{a}{e} + y$.

Equation (4) then gives $x^2 + (y+ae)^2 = e^2\left(\frac{a}{e} + y\right)^2$

or $\quad x^2 + y^2(1-e^2) = a^2(1-e^2)$

or $\quad \dfrac{x^2}{x^2(1-e^2)} + \dfrac{y^2}{a^2} = 1$ or $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1.$

This could also be got from (6) simply by interchanging $x$ and $y$.

**Note 2.** If we put $x=0$ in (7), we get $y = \pm b$ which gives a geometrically meaning to $b$ viz., $b$ is half the length intercepted on the $y$-axis by the ellipse.

## 8·13. Shape of an Ellipse.

Putting $y=0$ in $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ we get $x = \pm a$ so that $x$-axis meets the ellipse in the points $(\pm a, 0)$. They are the points A′, A.

So also the $y$-axis meets the ellipse in the points $(0, \pm b)$. We call them B, B′

Writing $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ as $y \pm \dfrac{b}{a}\sqrt{a^2 - x^2}$ ... (1)

we see that if $x > a$ or $< -a$, $a^2 - x^2$ becomes—*ive*.

∴ $y$ would become imaginary for all such values, so that no part of the curve lies to the right of the line $x=a$ or to the left of the line $x = -a$.

For any value of $x$ such that $-a < x < a$, equation (1) gives two equal and opposite values of $y$ so that for all such values the curve is symmetrical w.r.t. the $x$-axis or AA′.

So also writing $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ as $x = \pm \dfrac{a}{b}\sqrt{b^2 - y^2}$ we see that no part of the curve lies above the line $y=b$ or below the line $y = -b$ and that the curve is symmetrical w.r.t. the $y$-axis or BB′.

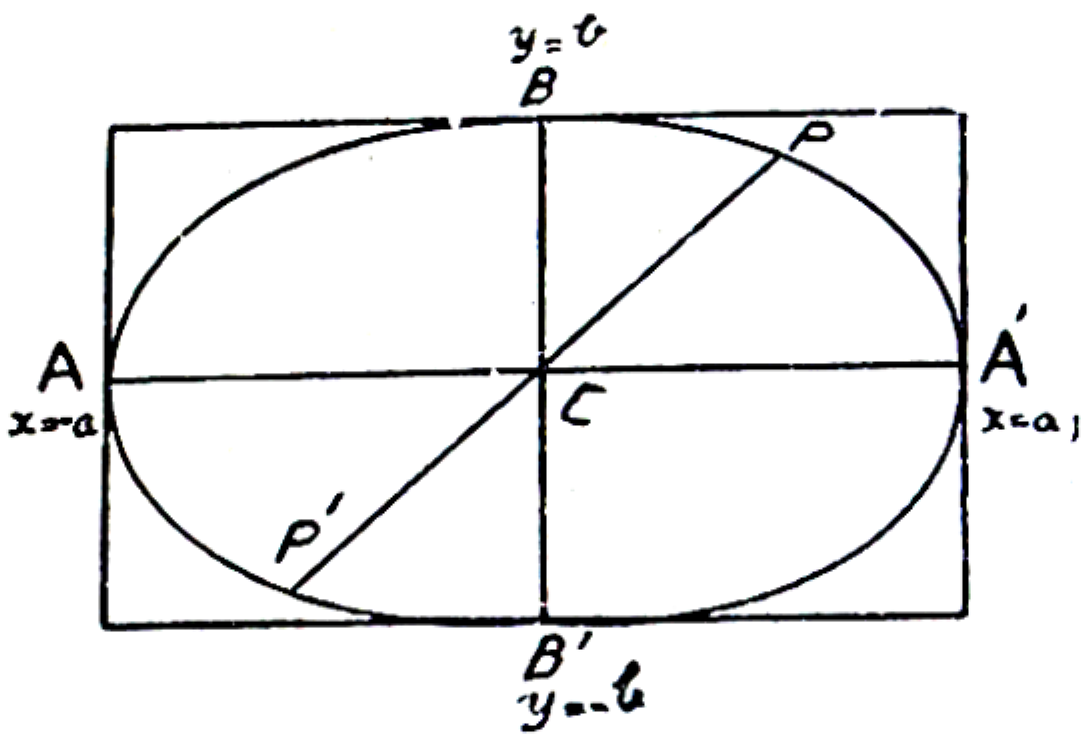
Hence the curve lies within the rectangle formed by the lines $y=\pm b$ $x=\pm a$.

Since there is symmetry about both axes, we need only see what happens in the first quadrant.

When $x=0$, equation (1) gives $y=b$.

As $x$ increases $y$ decreases and when $x=a$, $y=0$ so that as $x$ increases from zero to $a$, $y$ decreases from $b$ to zero.

Hence the ellipse has the form as shown in the figure.




**Definitions**

**Axes** :—AA′, BB′, the lines of symmetry for the ellipse, are known as the *Axes* of the ellipse and since AA′>BB′, AA′ is called the *Major* and BB′ the *Minor axis*.

Also AA′$=2a$, BB′$=2b$.

**The Centre** :—The point of intersection C of the axes is the centre of the ellipse.

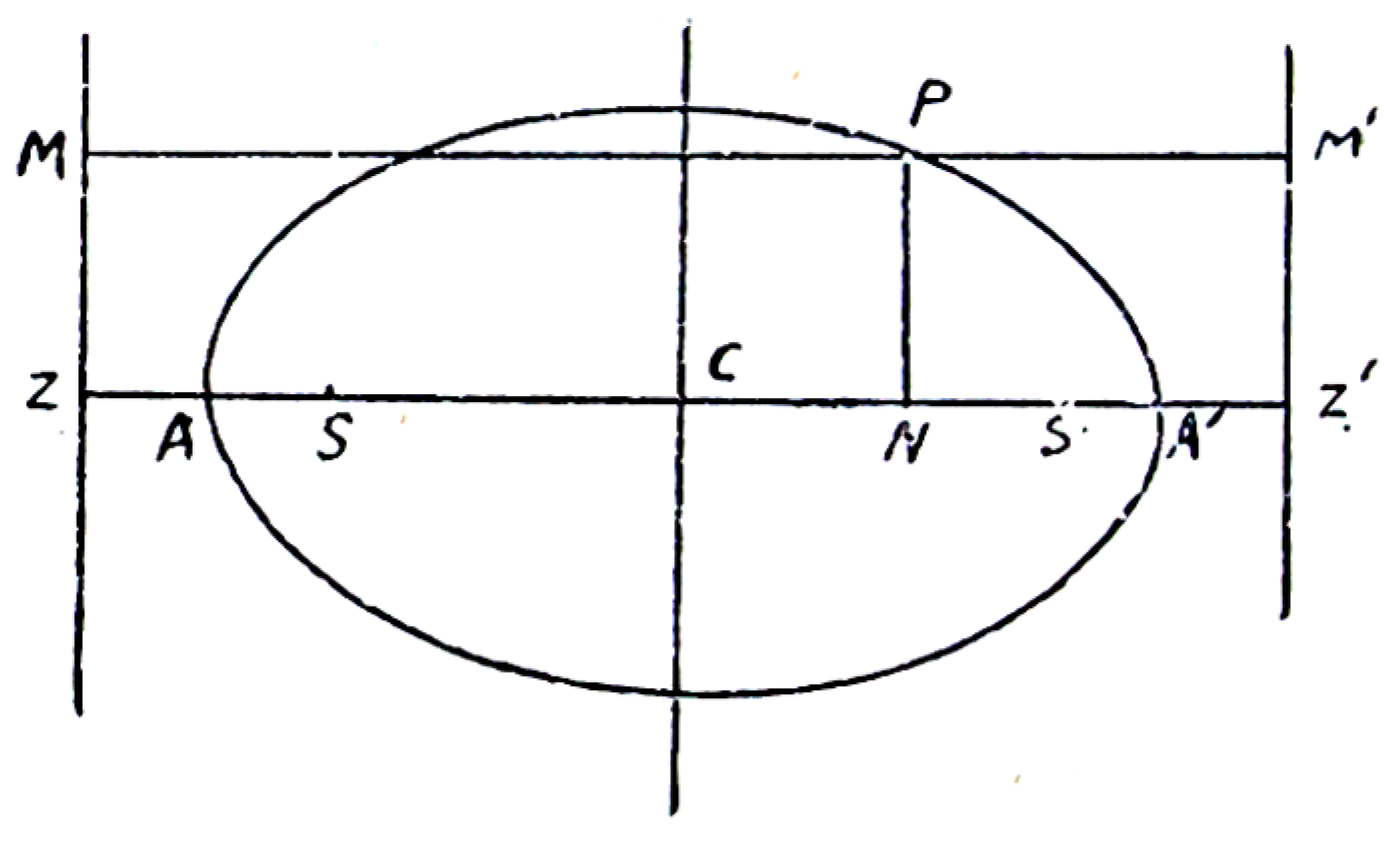
The centre bisects every chord of the ellipse through it.

For if P $(x_1, y_1)$ lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then it is obvious that $(-x_1, -y_1)$ also lies on the ellipse. The mid-point of $(x_1, y_1)$ and $(-x_1, -y_1)$ is $(0, 0)$ *i.e.*, the centre.

$\therefore$ The chord joining the points $(x_1, y_1)$ and $(-x_1, -y_1)$ on the ellipse is bisected at the centre.

**Vertices.** A, A′, B, B′ are called the vertices of the ellipse.

**8·4.** *The ellipse has a second focus and a corresponding second directrix.*




On the $+ive$ side of the $x$-axis take a point S′ and another Z′ such that SC$=$CS′$=ae$ and

ZC$=$CZ′ $=\frac{a}{e}$.

Draw Z′M′ $\parallel$ to ZM and PM′ $\perp$ Z′M′. Then the (equation (5) of Art. 8.12

*viz.*, $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$ may be written as

$$x^2+2aex+a^2e^2+y^2=e^2x^2+2aex+a^2$$

or $$x^2-2aex+a^2e^2+y^2=e^2x^2-2aex+a^2$$

or $$(x-ae)^3+y^2=e^2\left(x^2-2\frac{a}{e}x+\frac{a^2}{e^2}\right)=e^2\left(\frac{a}{e}-x\right)^2 \dots (1)$$

Since $PM'=NZ'=CZ'-CN=\frac{a}{e}-x$,

equation (1) can be interpreted as showing that the point $(x, y)$ moves so that its distance from the fixed point S′ $(ae, 0)$ bears a constant ratio '$e$' to its distance from the fixed line M′Z′.

Hence the ellipse could be generated equally well with S′ for a focus and M′Z′ for a directrix.

**Note** We could deduce the same result from considerations of symmetry of the ellipse about the $y$-axis.

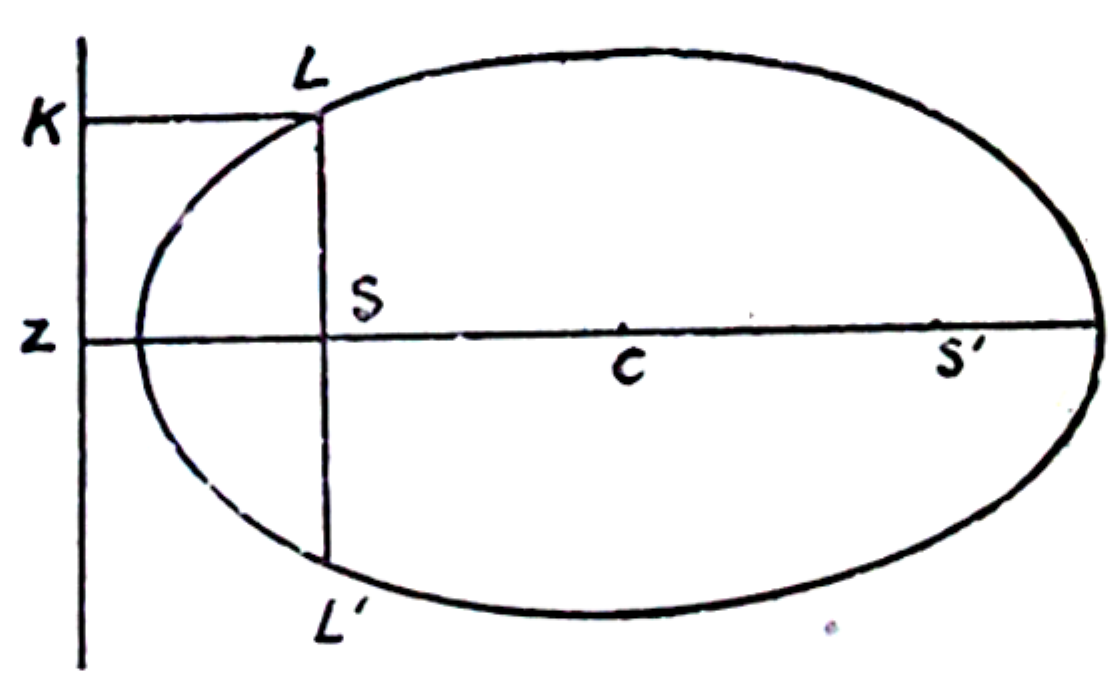
**8·15.** *The sum of the focal distances of any point on the ellipses is constant and equal to the major axis.*

$$SP=e.PM=e.ZN=e(ZC+CN)=e\left(\frac{a}{e}+x\right)=a+ex.$$

So also $S'P=e.PM'=e.NZ'=e(CZ'-CN)=e\left(\frac{a}{e}-x\right)$

$$=a-ex$$

Adding, $SP+S'T=2a$.




**8·16. Latus Rectum.** The chord of the ellipse passing through a focus and perpendicular to the major axis is called a latus rectum. The two together are called Latera Recta.

Let LSL′ be the latus rectum ; then

$$SL=e.LK=e.ZS=e(CZ-CS)=e\left(\frac{a}{e}-ae\right)=a-ae^2.$$

$$SL = a(1-e^2) = \frac{a^2(1-e^2)}{a} = \frac{b^2}{a}.$$

$\therefore$ The length of the latus rectum$=2SL=\dfrac{2b^2}{a}$

[We could get it also by supposing L to be $(-ae,\ y)$ then from $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,

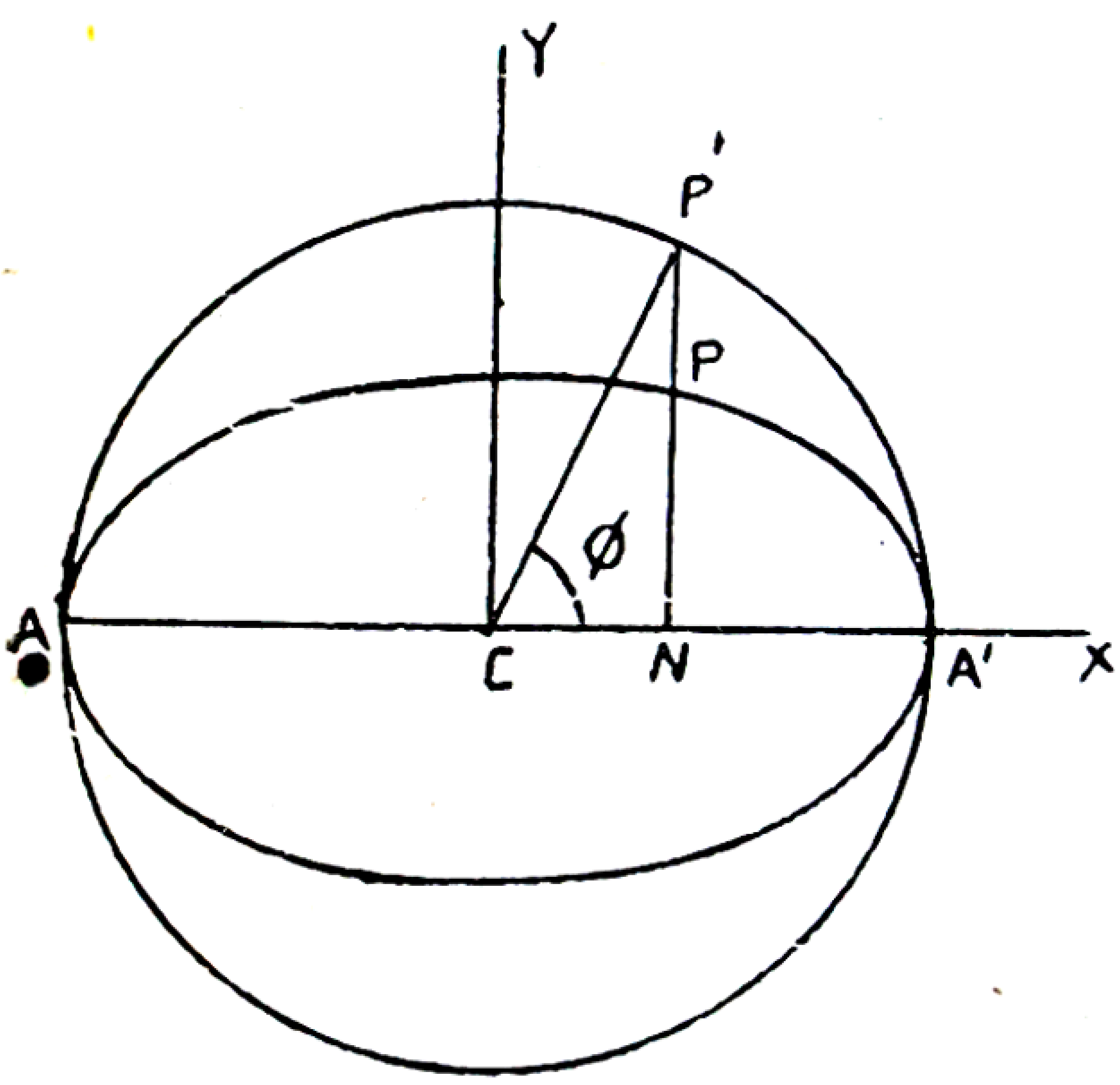
we have $SL=y=\sqrt{b^2(1-e^2)}=\dfrac{\sqrt{a^2a^2(1-e^2)}}{a}=\dfrac{\sqrt{b^4}}{a}=\dfrac{b^2}{a}$.]

**8·7. Def.** The circle on the major axis of an ellipse as diameter is called the **Auxiliary Circle**.

**8·18. Parametric Representation of a point on the Ellipse.**

Let $A'pA'$ be the auxiliary circle of the ellipse.

Let P be any point on the ellipse. Let the ordinate NP of P meet this circle in P′. Then the point P′ on the circle is said to correspond to the pt. P on the ellipse.




Join CP′ and let $\angle A'CP'$ be denoted by $\phi$.

Then the co-ordinates of $p'$ are (CN, NP′) and since $Cp'=a$. P′ is $(a\cos\phi, a\sin\phi)$

Co-ordinates of P are CN and NP, *i.e.*, $a\cos\phi$ and NP.

Let NP be $y$. Then $(a\cos\phi,\ y)$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$\therefore$ $y=b\sin\phi$ $\qquad\therefore$ P is $(a\cos\phi,\ b\sin\phi)$

which is the required parametric representation of any point P on the ellipse.

So for any pt. P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$x=a\cos\phi,\ y=b\sin\phi$

which latter are therefore called the parametric equations of the ellipse. As $\phi$ changes we get all the points on the ellipse.

**Def.** The angle $\phi$ is called the Eccentric Angle of the point P. We shall often refer to the point $(a\cos\phi,\ b\sin\phi)$ as the point '$\phi$' simply.

**8·19.** *To find the equation to the line joining the points '$\phi$' and '$\phi_1$' on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation is

$$\frac{x-a\cos\phi}{a(\cos\phi-\cos\phi_1)}=\frac{y-b\sin\phi}{b(\sin\phi-\sin\phi_1)}$$

or $$\frac{x-a\cos\phi}{-2a\sin\frac{\phi+\phi_1}{2}\sin\frac{\phi-\phi_1}{2}}=\frac{y-b\sin\phi}{2b\cos\frac{\phi+\phi_1}{2}\sin\frac{\phi-\phi_1}{2}}$$

or $$\frac{x-a\cos\phi}{-a\sin\frac{\phi+\phi_1}{2}}=\frac{y-b\sin\phi}{b\cos\frac{\phi+\phi_1}{2}}$$

or $$\frac{x}{a}\cos\frac{\phi+\phi_1}{2}+\frac{y}{b}\sin\frac{\phi+\phi_1}{2}=\cos\phi\cos\frac{\phi+\phi_1}{2}+\sin\phi\sin\frac{\phi+\phi_1}{2}$$

$$=\cos\left(\phi-\frac{\phi+\phi_1}{2}\right)$$

$$=\cos\left(\frac{\phi-\phi_1}{2}\right) \qquad \text{...(A)}$$

This equation is important and should be remembered.

**Cor.** To find the equation of the tangent at $\phi_1$ put $\phi_1=\phi$ in the above. We have

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

as the equation of the tangent at $\phi$

**Example.** *Find the equation to the chord of the ellipse joining the points whose eccentric angles are $\phi$ and $\phi+90°$.*

## Exercises VIII (A)

1. Find the equation of the ellipse referred to its principal axes as the axes of co-ordinates

(*i*) whose major axis$=10$ and minor axis$=6$.

(*ii*) whose major axis$=8$ and eccentricity$=\frac{1}{2}$.

(*iii*) whose minor axis$=10$ and eccentricity $=\frac{1}{\sqrt{2}}$.

(*iv*) whose foci are the points $(4, 0)$ and $(-4, 0)$ and eccentricity$=\frac{1}{4}$.

(*v*) whoss major axis$=18$ and the distance between the foci$=6$.

(*vi*) whose latus rectum$=\frac{8}{9}$ and eccentricity$=\frac{\sqrt{5}}{3}$.

(*vii*) whose sum of the axes$=54$ and the distance between the foci is$=18$.

2. Find the eccentricity of an ellipse

(*i*) whose major axis is double of the minor axis.

(*ii*) whose latus rectum is one third of the major axis.

(*iii*) the distance between whose foci is equal to the minor axis.

3. Find the eccentricity, foci, directrices, lengths and equations of the latera recta of the ellipses

(*a*) $4x^2+9y^2=36$, (*b*) $9x^2+4y^2=36$.

(*a*) may be written as $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\therefore\ a^2=9, \qquad b^2=4.$

Since $b^2=a^2(1-e^2)$, $4=9(1-e^2)$

$\therefore\ \frac{4}{9}=1-e^2 \qquad \therefore\ e^2=1-\frac{4}{9}=\frac{5}{9}$

$\therefore\ e=\frac{\sqrt{5}}{3}.$

Since $ae=3.\dfrac{\sqrt{5}}{3}=\sqrt{5}$ $\therefore$ foci are $(\pm ae, 0)$ or $(\pm\sqrt{5}, 0)$.

Again $\dfrac{a}{e}=3.\dfrac{3}{\sqrt{5}}=\dfrac{9}{\sqrt{5}}$.

The directrices *viz.*, $x=\pm\dfrac{a}{e}$ or $x=\pm\dfrac{9}{\sqrt{5}}$

Length of either latus rectum$=\dfrac{2b^2}{a}=\dfrac{8}{3}$.

Also their equations are $x=\pm ae=\pm\sqrt{5}$.

Writing it as $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ and attending to what was said in note Art., 8·14 we see that this is an ellipse whose major axis lies along the $y$-axis, whose foci therefore lie on the $y$-axis and whose directrices and latus-rectum will be parallel to $x$-axis.

$a$, $b$, $e$ are the same as in ($a$), so also $\dfrac{b^2}{a}$ and therefore the latus rectum.

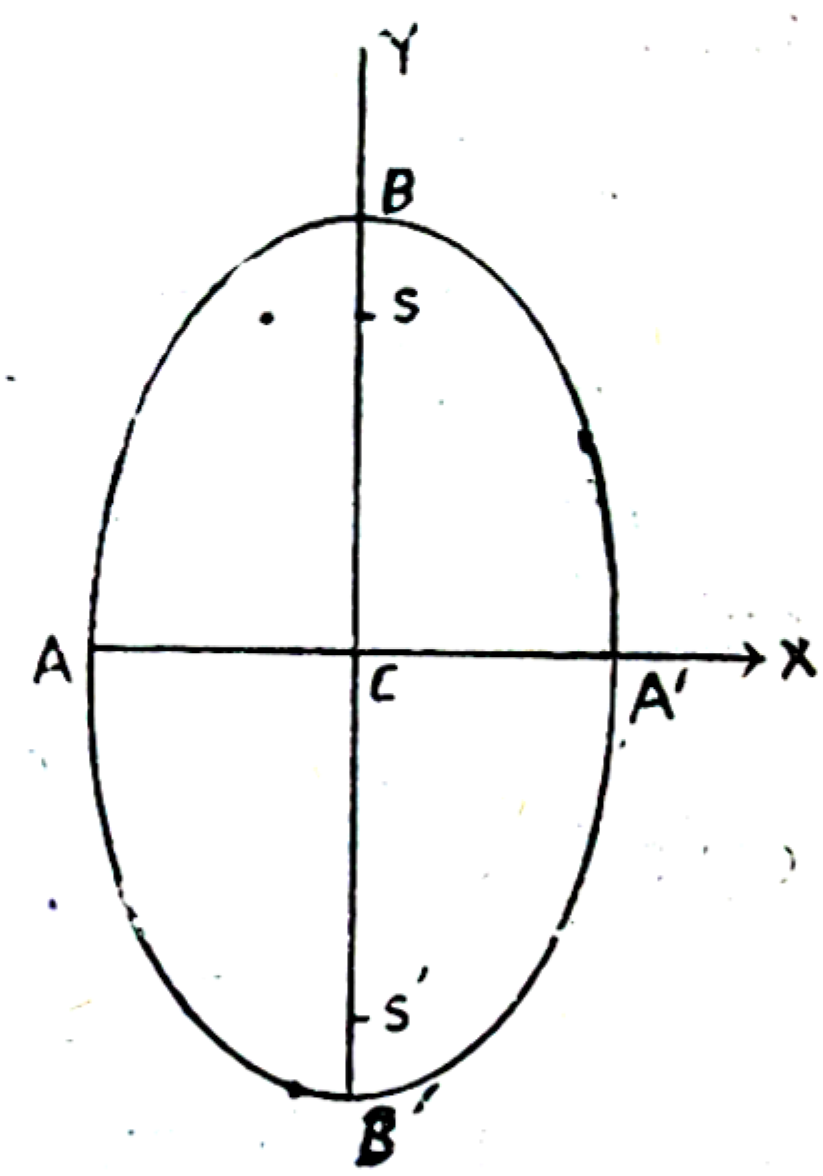
The foci are now $(0, +ae)$ or $(0, \pm\sqrt{5})$.

Equations to directrices are

$$y=\pm\frac{a}{e}$$
$$=\pm\frac{9}{\sqrt{5}}$$

and to latera recta.

$$y=\pm ae.$$
$$=\pm\sqrt{5}.$$




4. Find the eccentricity, latus rectum and the co-ordinates of the foci of the ellipses given by

(i) $5x^2+2y^2=1$. (ii) $x^2+3y^2=a^2$.

(iii) $9x^2+16y^2=144$. [P. U. 1936]

(iv) $9x^2+4y^2=36$.

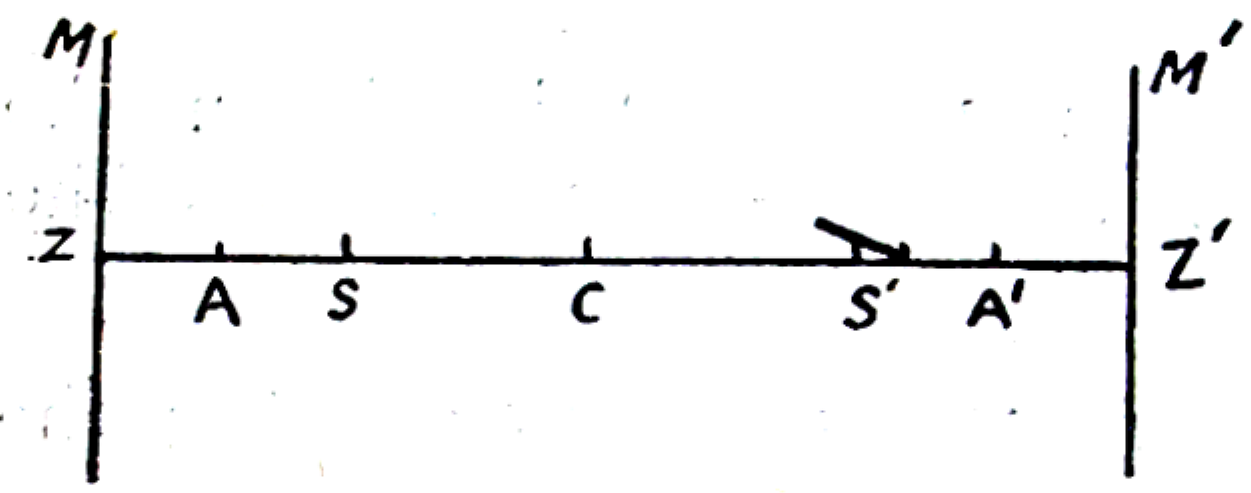
(v) $4x^2+3y^2=24$. [P. U.]

5. Find (*a*) the eccentricity and the distance between the foci of the ellipse $3x^2+4y^2=12$.

(*b*) the foci and the ends of the latera recta of the ellipse $2x^2+3y^2=1$.

6. Find the centre, the second focus, the second directrix of the ellipse whose one focus is $(2, -3)$ the corresponding directrix is $x+y+11=0$ and eccentricity$=\frac{2}{3}$.

[S is $(2, -3)$ and MZ is given on $x+y+11=0$.




$\therefore$ equation to SZ is $x-2-(y+3)=0$.

or $x-y-5=0$.

$\therefore$ Z, the point of intersection of SZ and MZ is $(-3, -8)$.

The vertices A, A′ divide SZ internally and externally in the ratio of 2 : 3.

$$\therefore \quad x \text{ of } A=\frac{-6+6}{5}=0 \text{ and } y \text{ of } A=\frac{-16-9}{5}=-5.$$

$\therefore$ A is $(0, -5)$.

$$x \text{ of } A'=\frac{-6-6}{-1}=12 \text{ and } y=\frac{-16+9}{-1}=7.$$

$\therefore$ A′ is $(12, 7)$.

$\therefore$ C the centre which is the mid-point of AA′ is $(6, 1)$.

Let the co-ordinates of S′ the second focus be $(x, y)$.

Then $(6, 1)$ is the mid-point of the join of S $(2, -3)$ and S′ $(x, y)$. $\therefore$ $\frac{x+2}{2}=6$ and $\frac{y-3}{2}=1$ giving S′ as $(10, 5)$.

Again if Z′ be $(x, y)$. Then C $(6, 1)$ is the mid-point of the join Z $(-3, -8)$ and Z′ $(x, y)$ whence Z′ is $(15, 10)$.

$\therefore$ equation to M′Z′ which is parallel to MZ is

$x-15+y-10=0$ or $x+y=25$

7. Find the centre, the second focus and directrix for the ellipse whose one focus is $(1, 2)$, the corresponding directrix is the line $x-y=5$ and eccentricity $\frac{1}{2}$. Also find the equation to that ellipse.

8. Show that the point $\left(\frac{a(1-t^2)}{1+t^2}, \frac{2bt}{1+t^2}\right)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of $t$.

9. Show that in an ellipse

(*i*) $\frac{PN^2}{AN.A'N}=\frac{BC^2}{CA^2}$

(*ii*) $AS.\ AS'=BC^2$

where PN is the perpendicular from any point P on the ellipse to the major axis AA′, C is the centre, and BB′ is the minor axis.

10. What are the values of the parameter $p$ for the vertices A′, B, A, B′ ?

11. Find the eccentric angle of the pt. (4, 3) on the ellipse

$$\frac{x^2}{64}+\frac{y^2}{12}=1.$$

12. What are the values of $p$ for points of intersection of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $bx=ay$ ?

13. Find the eccentric angle of a pt. on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre is 2.

14. Find the co-ordinates of the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are ($a$) $\frac{\pi}{4}$ ($b$) $\frac{\pi}{3}$.

15. If P and Q are the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are $\phi$ and $\frac{\pi}{2}+\phi$, then show that

$$CP^2+CQ^2=a^2+b^2.$$

16. P is the point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose

eccentric angle is $\phi$; show that $S'P=a(1-e\cos\phi)$ and $SP=a(1+e\cos\phi)$ where S and S′ are the foci.

**8·2.** The reader must, by this time, have become familiar with the methods that have been followed twice over; first for the circle and then for the parabola. He would not, therefore, require us to go into detail so far as the next few articles are concerned.

**8·21. Points of intersection of a line and an ellipse.**

The abscissae of the points of intersection of the straight line, $y=mx+c$ ......(1)

and the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ......(2)

are given by the quadratic $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1$

or $(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$ ...(3)

and the points would be real, coincident or imaginary depending on the nature of the roots of (3).

## Condition of Tangency

**Cor.** For tangency equation (3) above must have equal roots *viz.*,

$$a^4m^2c^2+a^2(a^2m^2+b^2)(c^2-b^2)=0$$

or $$a^2m^2b^2-b^2c^2+b^4=0$$

which gives $c^2=a^2m^2+b^2$ or $c=\pm\sqrt{a^2m^2+b^2}$ as the condition required.

Substituting for $c$ in (1), we get

$$y=mx\pm\sqrt{a^2m^2+b^2}. \qquad \text{... A)}$$

Whatever $m$, (A) will touch the ellipse. Also it is clear from (A) that two tangents can be drawn to an ellipse parallel to a given line or, to put it differently, in a given direction.

Equation (A) is important.

**8·22. Length of the chord intercepted on the line $y=mx+c$ by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.**

If $x_1, x_2$ are the roots of (3) of Art. 8·21,

$$\text{we have } x_1+x_2=\frac{-2a^2mc}{a^2m^2+b^2}$$

$$\text{and} \qquad x_1x_2=\frac{a^2(c^2-b^2)}{a^2m^2+b^2}.$$

$$\therefore \quad (x_1-x_2)^2=\frac{4a^2[m^2c^2a^2-(a^2m^2+b^2)(c^2-b^2)]}{(a^2m^2+b^2)^2}.$$

$$=\frac{4a^2b^2(a^2m^2+b^2-c^2)}{(a^2m^2+b^2)^2}.$$

So that the length of the chord

$$=\sqrt{(1+m^2)(x_1-x_2)^2}$$

$$=\frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}$$

**Cor.** Once again, for tangency, this length must be zero; whence $c^2=a^2m^3+b^2$

so that $c=\pm\sqrt{a^2m^2+b^2}$ as before.

**Example 1.** *Find the points of intersection of the line $3x+2y-6=0$ with the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$.*

Also find the length of the chord intercepted.

Solving together, we have

$$9x^2+(3x-6)^2=36 \quad \text{or} \quad 18x^2-36x=0$$

giving $x=0$ and $x=2$.

The corresponding values of $y$ are given by the equation of the st. line to be 3 and zero, so that the points of intersection are (0, 3) and (2, 0).

$\therefore$ Length of the intercept$=\sqrt{4+9}=\sqrt{13}$.

**Example 2.** *Find the equations of the tangents to the ellipse $4x^2+9y^2=18$ perpendicular to the line $3x-2y+1=0$.*

We have seen that $y=mx+\sqrt{a^2m^2+b^2}$ ......(1) touches the ellipse for all $m$. Writing the ellipse in the form $\dfrac{x^2}{\frac{9}{2}}+\dfrac{y^2}{2}=1$, we have $a^2=\frac{9}{2}$, $b^2=2$.

Also, the slope of the line being $\frac{3}{2}$, the slope of the tangents would be $-\frac{2}{3}$.

$\therefore$ from (1), the tangents are $y=-\frac{2}{3}x\pm\sqrt{\frac{9}{2}\cdot\frac{4}{9}+2}$.

or $2x+3y\pm6=0$.

Aliter Any line perpendicular to $3x-2y+1=0$ is

$2x+3y+k=0$ ......(2)

Solving with the ellipse, we have $4x^2+\frac{9(k+2x)^2}{9}=18$

or $8x^2+4kx+(k^2-18)=0$ ......(3)

For (2) to touch the ellipse, (3) must have equal roots.

$\therefore$ $16k^2-32(k^2-18)=0$, giving $k=\pm6$

$\therefore$ Tangents are $2x+3y\pm6=0$.

### 8·23. Chord and Tangent.

(*a*) *To find the equation of the chord joining* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\frac{x}{a^2}(x_1+x_2)+\frac{y}{b^2}(y_1+y_2)=\frac{x_1}{a^2}(x_1+x_2)+\frac{y_1}{b^2}(y_1+y_2)$$

$$=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+1 \text{......(A) [by (1)].}$$

**Cor.** The equation of the chord joining the points '$\phi$' and '$\phi_1$' may be obtained from equation (A) by making $x_1=a\cos\phi$, $y_1=b\sin\phi$, $x_2=a\cos\phi_1$ and $y_2=b\sin\phi_1$. Making these substitutions we get

$$\frac{x}{a}(\cos\phi+\cos\phi_1)+\frac{y}{b}(\sin\phi+\sin\phi_1)$$

$$=\cos\phi\cos\phi_1+\sin\phi\sin\phi_1+1$$

or $$\frac{2x}{a}\cos\frac{\phi+\phi_1}{2}\cos\frac{\phi-\phi_1}{2}+\frac{2y}{b}\sin\frac{\phi+\phi_1}{2}\cos\frac{\phi-\phi_1}{2}$$

$$=1+\cos(\phi-\phi_1)$$

$$=2\cos^2\frac{\phi-\phi_1}{2}$$

or $$\frac{x}{a}\cos\frac{\phi+\phi_1}{2}+\frac{y}{b}\sin\frac{\phi+\phi_1}{2}=\cos\frac{\phi-\phi_1}{2} \quad \text{......(A}')$$

[*Cf* Art. 8·19]

Both P $(x_1, y_1)$ and Q $(x_2, y_2)$ lie on the ellipse

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1, \quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \quad \text{......(1)}$$

or $$\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0$$

or $$\frac{y_2-y_1}{x_2-x_1}=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1} \quad \text{......(2)}$$

Equation to PQ is $y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$

Equation to PQ is $y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$

or $$y-y_1=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}(x-x_1) \quad \text{......(3) [by (2)]}$$

(*b*) *Tangent at P* $(x_1, y_1)$ :—

PQ would be a tangent to the ellipse if Q coincides with P. Making $x_2=x_1$ and $y_2=y_1$ in (3) we have for the equation of the tangent

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{x_1}{y_1}(x-x_1) \quad \text{......(4)}$$

or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \text{......(B)}$$

[by the virtue of (1) above]

**Cor.** Tangent at $(a\cos\phi, b\sin\phi)$ is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1 \text{......(B}') \quad \text{[}Cf\text{ Art. 8·19 Cor.]}$$

(*c*) *To find the equation of the normal at P* $(x_1, y_1)$ *to the ellipse.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The normal at $(x_1\ y_1)$, being a line perpendicular to the tangent at $(x_1, y_1)$ has for its equation

$$y-y_1=\frac{a^2}{b^2}\cdot\frac{y_1}{x_1}(x-x_1)$$

or
$$\frac{a^2}{x_1}(x-x_1)=\frac{b^2}{y_1}(y-y_1) \qquad \text{......(C)}$$

**Cor.** Normal at $(a \cos \phi, b \sin \phi)$ is

$$\frac{a \cos \phi}{\cos \phi}=\frac{b(y-b \sin \phi)}{\sin \phi}$$

or
$$\frac{ax}{\cos \phi}-\frac{by}{\sin \phi}=a^2-b^2 \text{........(C}')$$

*Note.* Equations (A), (A′), (B), (B′), (C), (C′) shall be of frequent use and should, therefore, be remembered.

**Example 1.** *Obtain the equations of the tangent and normal to $x^2+5y^2=14$ at $(3, 1)$.*

Substituting in the equation of the tangent
$xx_1+5yy_1=14$, we get $3x+5y=14$.

Slope of the tangent $=-\frac{3}{5}$

$\therefore$ ,, ,, ,, normal $=\frac{5}{3}$.

$\therefore$ Normal at $(3, 1)$ is $y-1=\frac{5}{3}(x-3)$
or $5x-3y=12$.

**8·24.** If a line is given to be a tangent to an ellipse, the equations of the line and the curve solved simultaneously must give a perfect square in one of the variables. That would give us one of the co-ordinates of the point of contact; the other could be found from the equation of the straight line.

**Example 2.** *Show that $2x+y-4=0$ touches the ellipse $4x^2+y^2=8$. Also find the point of contact.*

Solving together, $4x^2+(2x-4)^2=8$.

giving $\quad 8(x-1)^2=0$.

(*i*) a perfect square, showing that the line touches the curve.

Also (*i*) gives $x=1$ and from the equation of the line, we get $y=2$ so that $(1, 2)$ is the point of contact.

The following method, however, is much more convenient in practice, especially where we have to deal with equations with algebraic co-efficients.

**Example 3.** *Find the condition of tangency of the line* $lx+my+n=0$ *with the ellipse.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.\quad \text{Also find the point of contact.}$$

Let $(x_1, y_1)$ be the point of contact.

$(x_1, y_1)$ lies on $lx+my+n=0$.

$$\therefore \quad lx_1+my_1+n=0 \qquad \ldots(i)$$

Tangent at $(x_1, y_1)$ to the ellipse is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

which will be the same as $lx+my+n=0$

$$\text{if} \quad \frac{x_1}{a^2 l}=\frac{y_1}{b^2 m}=-\frac{1}{n}$$

$$\text{giving } x_1=-\frac{a^2 l}{n} \text{ and } y_1=-\frac{b^2 m}{n}.$$

*as the point of contact.*

Substituting in (*i*) we get $\frac{a^2l^2}{n}+\frac{b^2m^2}{n}-n=0$

or $\quad a^2l^2+b^2m^2-n^2=0$ as the condition required.

**8·25. Equation of a chord in terms of the co-ordinates of the middle point.**

If M $(h, k)$ be the middle points of the chord PQ whose slope is $m$, the equation to it is

$$y-k=M(x-h) \qquad \ldots(1)$$

Also if P be $(x_1, y_1)$ and Q $(x_2, y_2)$, the equation to PQ is

$$(x_1+x_2)\frac{x}{a^2}+(y_1+y_2)\frac{y}{b^2}=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+1 \qquad \ldots(2)$$

[Eqn. (A) Art. 8·23]

$$\therefore \quad m=-\frac{x_1+x_2}{y_1+y_2}\cdot\frac{b^2}{a^2} \qquad \ldots(3)$$

Since $x_1+x_2=2h$ and $y_1+y_2=2k$,

Eqn. (3) gives $m=-\dfrac{h}{k}\cdot\dfrac{b^2}{a^2}$ so that (1) becomes

$$y-k=-\frac{h}{k}\cdot\frac{h^2}{a^2}(x-h)$$

or $$\frac{h}{a^2}(x-h)+\frac{k}{b^2}(y-k)=0$$

or $$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}.$$

*Note.* If the mid-point be $(x_1, y_1)$ the equation will be

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

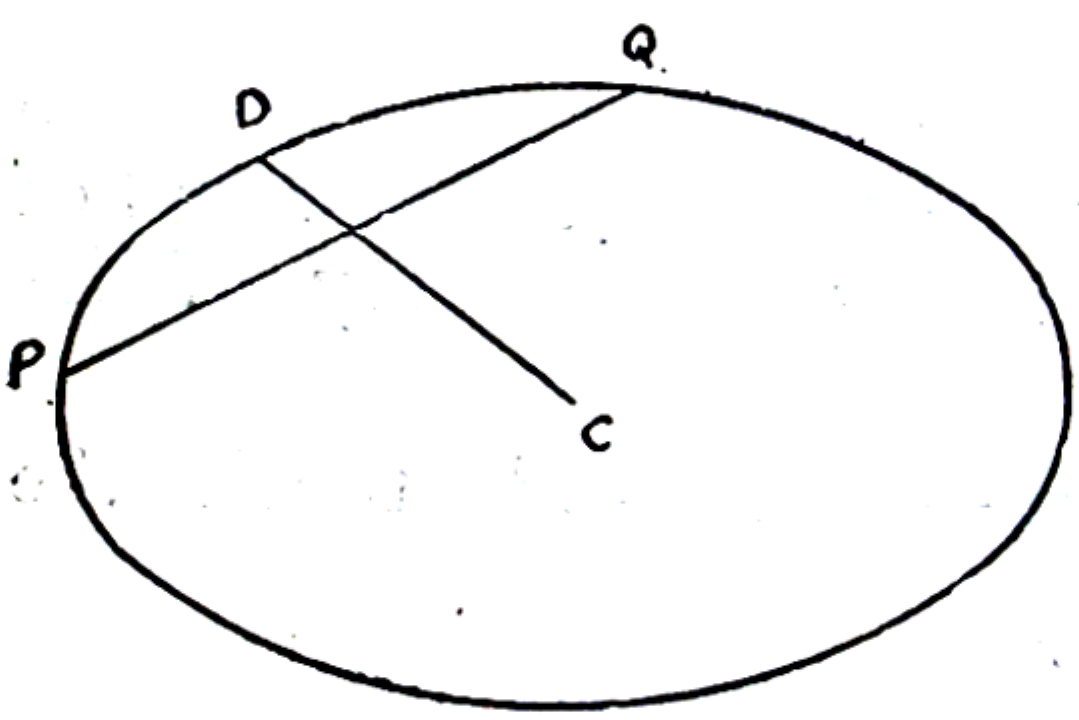
**Example.** *Find the equation of the chord of the ellipse* $\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$ *which is bisected at the point* (2, 3).

Making $x_1=2$, $y_1=3$ and $a^2=4$, $b^2=3$ in the equation of the note, we get $\dfrac{2x}{4}+\dfrac{3y}{3}=\dfrac{4}{4}+\dfrac{9}{3}$.

or $x+2y-8=0$.

**8·26. Locus of the middle-points of a system of parallel chords.**

If '$m$' be the fixed slope of the parallel system, equation (3) above gives




$$m=-\frac{x_1+x}{y_1+y_2}\cdot\frac{b^2}{a^2}=-\frac{h}{k}\cdot\frac{b^2}{a^2}$$

so that the middle-point $(h, k)$ lies on the locus whose equation is

$$-\frac{x}{y}\cdot\frac{b^2}{a^2}=m$$

or $$y=-\frac{b^2}{a^2m}x \qquad \ldots(1)$$

which is a straight line passing through the centre of the ellipse.

This locus again is called a *diameter* of the ellipse.

**Cor.** Since $y=mx$ is the diameter parallel to the system of parallel chords equation (1) states that the diameter $y=m'x$ will bisect chords parallel to the diameter $y=mx$ if $m'=\dfrac{b^2}{a^2m}$.

## Exercises VIII (B)

1. (*a*) Find the length of the chord intercepted by the ellipse $2x^2+3y^2=10$ on the line $y=x$.

(*b*) Find the length of the chord intercepted by the ellipse $x^2+3y^2=4$ on the line $x+y=2$.

2. Show that $x+2y-4=0$ is a tangent to $3x^2+4y^2=12$ ; also find the point of contact.

3. Find the tangents to the ellipse $4x^2+3y^2=5$ which are parallel to the line $y=3x+7$.

4. Find the tangents to the ellipse $4x^2+3y^2=1$ which are $\perp$ to the line $3x-4y+5=0$.

5. Find the tangents to the ellipse $b^2x^2+a^2y^2=a^2b^2$ which make equal intercepts on the two axes.

6. Find the equation to the tangent and normal to the ellipse $4x^2+3y^2=24$ at the point $(\sqrt{3}, 2)$.

7. (*a*) Prove that the tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the extremities of a latus-rectum which is in the first quadrant, the corresponding directrix and the major axis meet in a point.

(*b*) Write down the equation to the normal at the same point and prove that it shall pass through the lower end of the minor axis if $e^4+e^2=1$, where $e$ is the eccentricity of the ellipse.

8. Show that the line $lx+my+n=0$ will touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if $a^2l^2+b^2m^2=n^2$

9. Show that the line $x \cos \alpha+y \sin \alpha=p$. ... (1)

will touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ... (2)

if $a\cos^2 a+b^2\sin^2\alpha=p^2$.

[In addition to the methods already given, we can also proceed as under :—

Let (1) touch (2) at the point ($a\cos\phi, b\sin\phi$), then (1) must be the same as $\dfrac{x\cos\phi}{a}+\dfrac{y\sin\phi}{b}=1$. Comparing co-efficients, $\dfrac{\cos\phi}{a\cos\alpha}+\dfrac{\sin\phi}{b\sin\alpha}=\dfrac{1}{p}$.

To obtain the required condition we have to eliminate $\phi$ from relations (3). Now $\cos\phi=\dfrac{a\cos\alpha}{p}$; $\sin\phi=\dfrac{b\sin\alpha}{p}$.

Squaring and adding we get $a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$.

*Note.* Equation (3) also gives the point of contact [$a\cos\phi, b\sin\phi$]]

10. Show that the tangents at the corresponding points to the ellipse and the auxiliary circle intersect on the major axis of the ellipse.

11. Tangents at any point P of an ellipse meet the axes in Q,R. The rectangle CQTR is completed. Show that the locus of T is the ellipse $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$.

[Let the tangent be $\dfrac{x\cos\phi}{a}+\dfrac{y\sin\phi}{b}=1$ ; then T is $x=\dfrac{a}{\cos\phi}$; $y=\dfrac{b}{\sin\phi}$. Eliminate $\phi$.]

12. Find the common tangents to the ellipse $\dfrac{x^2}{a^2+b^2}+\dfrac{y^2}{b^2}=1$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2+b^2}=1$.

[*Hint.* If $m$ be the slope of a common tangent, the equations $y=mx+\sqrt{(a^2+b^2)m^2+b^2}$ and $y=mx+\sqrt{a^2m^2+a^2+b^2}$ must be identical. Equating $(a^2+b^2)m^2+b^2$ to $a^2m^2+a^2+b_2$ we get $m=\pm\dfrac{a}{b}$, whence easily the tangents.]

13. Find the locus of the point of intersection of the lines $\dfrac{tx}{a}-\dfrac{y}{b}+t=0$ and $\dfrac{x}{a}+\dfrac{ty}{b}-1=0$, $t$ being the variable parameter.

14. Find the equation to the locus of a point which moves so that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is constant.

15. Any ordinate NP of an ellipse meets the auxiliary circle in Q. Prove that the normals at P and Q intersect on the circle $x^2+y^2=(a+b)^2$.

16. P, P′ and Q, Q′ are pairs of corresponding points on the ellipse and the auxiliary circle, prove that PP′, QQ′ meet the major axis in the same point.

17. Find the equation of the chord of the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$ which is bisected at $(1, 1)$.

18. Find the locus of the mid-points of the portion of tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ intercepted between the axes.

19. The normal at P meets the major axis in G. Find the locus of the mid-point of PG.

20. Find the locus of mid-points of the chords of an ellipse which subtend a rt. $\angle$ at the centre.

21. Find the locus of mid-points of chords of an ellipse

(*a*) which pass through A $(a, 0)$,

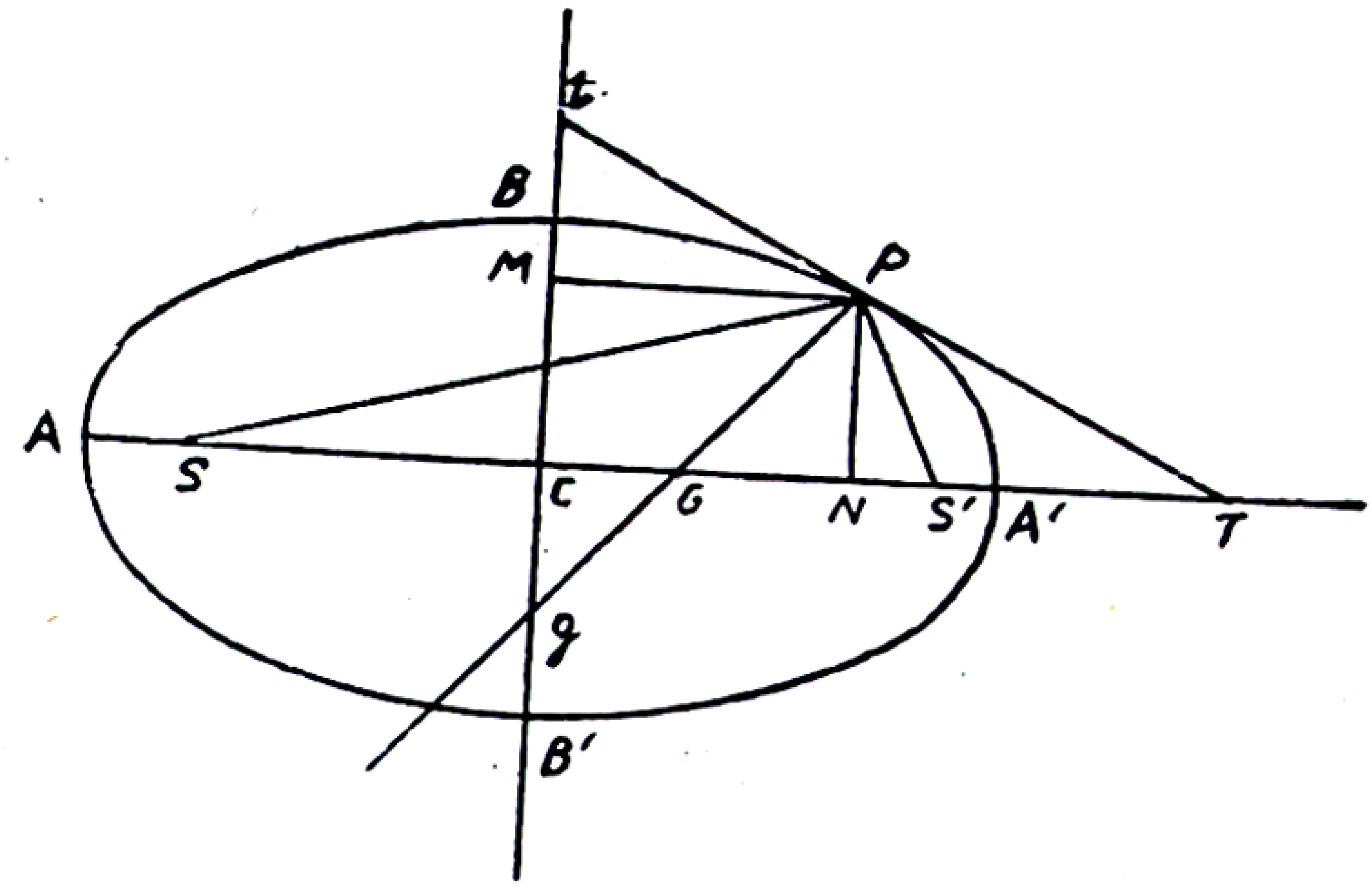
(*b*) which pass through the fixed point $(h, k)$.

## Geometrical Properties of the Ellipse

*Note 1.* The equation of the ellipse throughout this section will be taken as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

*Note 2.* The figure opposite will serve throughout the section




**8·31.** *The sum of the focal distances of any point on an ellipse is constant and equals its major axis.*

Let P $(a \cos, \phi\ b \sin \phi)$ be any point on the ellipse.

The foci S and S′ being $(-ae,\ 0)$, $(+ae,\ 0)$ respectively,

$$SP=\sqrt{(a \cos \phi+ae^2)+b^2 \sin^2 \phi}.$$

$$=\sqrt{(a^2 \cos^2 \phi+2a^2e \cos \phi+a^2e^2+a^2\sin^2\phi-a^2e^2 \sin^2\phi)}$$

$$=\sqrt{(a^2+2a^2e \cos \phi+a^2e^2 \cos^2\phi)}$$

$$=a+ae \cos \phi.$$

Similarly, $S'P=a-ae \cos \phi$

$\therefore\ SP+S'P=2a=$ the major axis. [*Cf.* Art. 8·15]

**8·32.** *The normal and the tangent bisect the interior and exterior angles respectively between the focal distances of the point.*

Let P be $(a \cos \phi,\ b \sin \phi)$

Then (Art. 8·31) $SP=a\ (1+e \cos \phi)$.

$S'P=a\ (1-e \cos \phi)$.

$$\therefore \quad \frac{SP}{S'P}=\frac{1+e \cos \phi}{1-e \cos \phi}. \qquad \ldots (1)$$

Equation of the normal at P is

$$a \sec \phi.\ x-b \operatorname{cosec} \phi.\ y=a^2-b^2. \qquad \ldots (2)$$

Where this meets the $x$-axis, $a \sec \phi.\ x=a^2-b^2.$

$$\therefore \quad x=\frac{a^2-b^2}{a \sec \phi}=\frac{a^2e^2}{a \sec \phi}=ae^2 \cos \phi.$$

Hence G is $(ae^2 \cos \phi, 0)$.

$$\therefore \quad \frac{SG}{SG'}=\frac{SC+CG}{CS'-CG}=\frac{ae+ae^2 \cos \phi}{ae-ae^2 \cos \phi}$$

$$=\frac{1+e \cos \phi}{1-e \cos \phi}=\frac{SP}{SP'}, \text{ from (1)}$$

$\therefore$ the normal PG bisects the angle SPS′ internally and the tangent, being perpendicular to the normal, therefore bisects it externally.

**8·33.** *The semi-minor axis of an ellipse is the mean proportional between the perpendiculars from the foci on any tangent of the ellipse.*

Let P $(a \cos \phi, b \sin \phi)$ be any point on the ellipse.

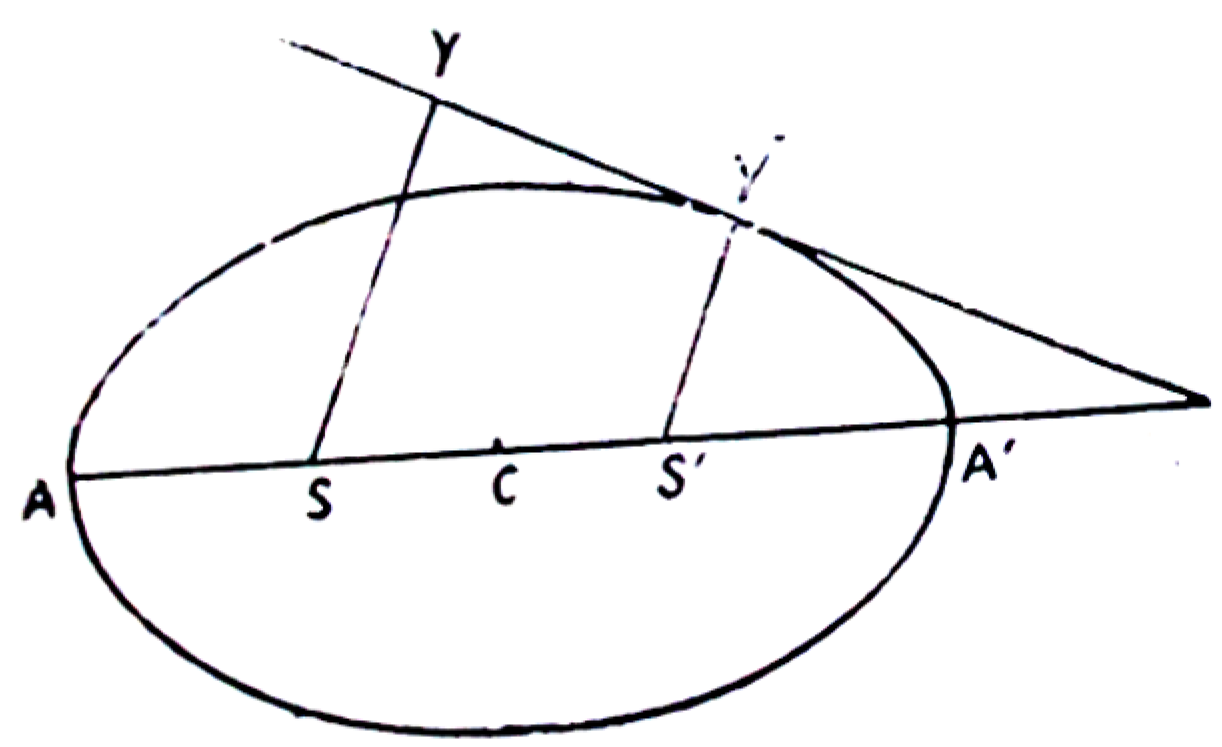
Tangent at P is $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1.$

or $bx \cos \phi + ay \sin \phi - ab = 0.$ ...(1)

The perpendicular SY from the focus S $(-ae, 0)$ on this tangent

$$=\frac{-abe \cos \phi - ab}{\sqrt{b^2 \cos^2 \phi + a^2 \sin^2 \phi}}$$

$$=\frac{-ab(1-e \cos \phi)}{\sqrt{b^2 \cos^2 \phi + a^2 \sin^2 \phi}}$$




Similarly

$$S'Y'=\frac{-ab(1-e \cos \phi)}{\sqrt{(b^2 \cos^2 \phi + a^2 \sin^2 \phi)}}$$

$$\therefore \quad SY.S'Y'=\frac{a^2b^2(1-e^2 \cos^2 \phi)}{b^2 \cos^2 \phi + a^2 \sin^2 \phi}=\frac{a^2b^2(1-e^2 \cos^2 \phi)}{a^2(1-e^2) \cos^2 \phi + a^2 \sin^2 \phi}$$

$$=\frac{b^2(1-e^2 \cos^2 \phi)}{1-e^2 \cos \phi^2}=b^2.$$

**8.34.** *The locus of the feet of the perpendiculars from the foci on any tangent is the auxiliary circle.*

The equation of the tangent at P $(a \cos \phi, b \sin \phi)$ is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1,$$

or $b \cos \phi . x + a \sin \phi . y = b$ ...(1)

The line through $S(-ae, 0)$ perpendicular to (1) is

$$a \sin \phi (x - ae) - b \cos \phi . y = 0$$

or $a \sin \phi . x - b \cos \phi . y = -a^2 e \sin \phi$ ...(2)

Squaring (1) and (2) and adding.

$$(x^2 + y^2)(a^2 \sin^2 \phi + b^2 \cos^2 \phi) = a^2 b^2 + a^2 e^2 \sin^2 \phi$$
$$= a^2(b^2 + a^2 e^2 \sin^2 \phi) = a^2[b^2 \cos^2 \phi + (a^2 e^2 - b^2) \sin^2 \phi]$$
$$= a^2(b^2 \cos^2 \phi + a^2 \sin^2 \phi).$$

Dividing both sides by $a^2 \sin^2 \phi + b^2 \cos^2 \phi$, we get

$x^2 + y^2 = a^2$ as the locus required.

*Note.* As an exercise, let the student take the equation of the tangent as $y = mx + \sqrt{a^2 m^2 + b^2}$ and then obtain the result.

**8·35.** *The locus of the point of intersection of perpendicular tangents to an ellipse is a circle.*

The line $y = mx + \sqrt{a^2 m^2 + b^2}$

or $y - mx = \sqrt{a^2 m^2 + b^2}$ ...(1)

is a tangent to the ellipse for all values of $m$.

For the tangent, perpendicular to it, slope $= -\dfrac{1}{m}$.

Hence the equation of the perpendicular tangent is

$$y = -\frac{1}{m} x + \sqrt{\frac{a^2}{m^2} + b^2}$$

or $my + x = \sqrt{a^2 + b^2 m^2}$. ...(2)

To obtain the equation of the locus of their point of intersection, we have to eliminate $m$ between (1) and (2). Squaring and adding them, we have

$$y^2 + m^2 x^2 + x^2 + m^2 y^2 = a^2 m^2 + b^2 + a^2 + b^2 m^2$$

or $$(1 + m^2)(x^2 + y^2) = (1 + m^2)(a^2 + b^2)$$

or $$x^2+y^2=a^2+b^2$$
as the equation of the required locus which is evidently a circle having its centre at the origin.

**Def.** This circle is called the **Director Circle** of the ellipse.

**Example 1.** *If the tangent at a point P on an ellipse meets the major axis in T and the minor axis in t and N and M are the feet of the perpendiculars from P on the major and minor axes respectively, show that*

(*i*) $CN.CT=a^2$ (*ii*) $CM.Ct=b^2$.

**Example 2.** *If the normal to an ellipse at a point P meets the major axis in G and N is the foot of the ordinate of P, show that* $CG=e^2.CN$ *where C is the centre of the ellipse.*

**Example 3.** *If the normal to an ellipse at a point P meets the major and minor axes in G and g respectively, and CF is the perpendicular from the centre C to this normal, show that* (*i*) $PF.PG=b^2$ *and* (*ii*) $PF.Pg=a^2$.

**Position of a point relative to an ellipse.**

Proceeding exactly as we did in the case of the parabola, the student should have no difficulty in proving that the pt. $S(x_1, y_1)$ will be outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 \lesseqgtr 0$.

## 8·41. Tangents from a point to an ellipse.

We have seen that $y=mx+\sqrt{a^2m^2+b^3}$ ...(1)

touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whatever $m$ may be.

It will pass through the point $(x_1, y_1)$ if

$y_1=mx_1+\sqrt{a^2m^2+b^2}$.

This equation which may be written as
$$(y_1-mx_1)^2=a^2m^2+b^2$$
or $$(x_1^2-a^2)m^2-2x_1y_1m+y_1^2-b^2=0 \quad \text{...(2)}$$
gives us the slopes of the tangents that pass through the pt,

$(x_1, y_1)$. Equation (2) being a quadratic in $m$, two tangents can be drawn to the ellipse from $(x_1, y_1)$. And they will be real, coincident, or imaginary according as the roots of (2) are real, coincident, or imaginary

*i.e.*, according as $x_1^2y_1^2-(x_1^2-a^2)(y_1^2-b^2) \lessgtr 0$

according as $b^2x_1^2+a^2y_1^2-a^2b^2>,=\text{or}<0$.

or according as $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}>,=\text{or}<1$

*i.e.*, according as the pt. $(x_1, y_1)$ is outside, on, or inside the ellipse.

*Note.* The equations to the two tangents from $(x_1, y_1)$ may be obtained by substituting in (1) the two values of $m$ got from (2).

**8·42.** *Locus of the point of intersection of tangents to an ellipse inclined at a constant angle $\alpha$.*

Let $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ be the ellipse and $(x_1, y_1)$ the point of intersection of the tangents.

$$y=mx+\sqrt{x^2m^2+b^2_1} \qquad \ldots\ldots(1)$$

touches the ellipse for all values of $m$.

(1) will pass through the point $(x_1, y_1)$ if

$$y_1=mx_1+\sqrt{a^2m^2+b^2} \qquad \ldots\ldots(2)$$

Equation (2) which may be re-written as

$$(y_1-mx_1)^2=a^2m^2+b^2.$$

or $$(x_1^2-a^2)m^2-2x_1y_1m+y_1^2-b^2=0 \qquad \ldots\ldots(3)$$

gives the slopes of the two tangents to the ellipse that pass the point $(x_1, y_1)$.

If $m_1$, $m_2$ be the roots of (3)

$$m_1+m_2=\frac{2x_1\,y_1}{x_1^2-a^2} \qquad \ldots\ldots(4)$$

$$m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2} \qquad \ldots\ldots(5)$$

By the question,

$$\tan \alpha = \frac{m_1 - m_2}{1+m_1m_2}$$

$$= \frac{\sqrt{(m_1+m_2)^2 - 5m_1m_2}}{1+m_1m_2}$$

$$= \frac{2\sqrt{\dfrac{x_1^2y_1^2}{(x_1^2-a^2)^2} - \dfrac{y_1^2-b^2}{x_1^2-a^2}}}{1+\dfrac{y_1^2-a^2}{x_1^2-a^2}}$$

$$= \frac{2\sqrt{x_1^2y_1^2 - (x_1^2-a^2)(y_1^2-b^2)}}{x_1^2+y_1^2-a^2-b^2}$$

$$= \frac{2\sqrt{b^2x_1^2+a^2y_1^2-a^2b^2}}{a_1^2+y_1^2-a^2-b^2}$$

$$= \frac{2ab\sqrt{\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1}}{x_1^2+y_1^2-a^2-b^2}$$

so that $(x_1, y_1)$ lies on the locus given by

$$(x^2+y^2-a^2-b^2)^2 = 4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\cot^2\alpha \qquad \ldots(6)$$

**Cor.** The particular case of intersection of $\perp$ tangents may be disposed of either by (5) which gives

$$m_1m_2 = \frac{y_1^2-b^2}{x_1^2-a^2} = -1$$

so that $x^2+y^2=a^2+b^2$ is the locus required ;

or by putting $\alpha=90'$ in (6). [Also see Q. 7].

**8·43.** The discussion of normals from a point and of foci arising out of it is too complicated for this booklet. We shall content ourselves with telling the reader that in general four normals can be drawn from a pt. to an ellipse. And this he can see for himself by putting $\tan\dfrac{\phi}{2}=t$ in the equation

$\dfrac{ax}{\cos\phi} - \dfrac{by}{\sin\phi} = a^2-b^2$.........(1) which is the normal to the

ellipse at any point '$\phi$' and arranging it as an equation of the fourth degree in $t$ which must have four roots.

## Exercises VIII (C)

1. Find the equations of the two tangents to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ from the pt. $(-15, -7)$ [See note to Art. 8 41]

2. Find the equations of the two tangents to the ellipse $x^2+2y^2=6$ from the point $(4, 1)$.

3. Prove that the angle between the two tangents from $(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

$$\tan\theta=\frac{2ab\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1}}{x_1^2+y_1^2-a^2-b^2}$$

Tangents are drawn from a point P to the ellipse making angles $\theta_1$ and $\theta_2$ with the major axis. Find the locus of P when

4. $\theta_1+\theta_2=$constant (say $2\alpha$).

5. $\tan\theta_1+\tan\theta_2=c$. 6. $\tan\theta_1-\tan\theta_2=d$.

7. $\tan^2\theta_1+\tan^2\theta_2=k$. 8. $\cot\theta_1+\cot\theta_2=g$.

**8·5. Chord of Contact.** As was proved in the case of the parabola, it can be shown that the equation of the chord of contact of tangents from $P(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\text{is}\quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

**8·51. Pole and Polar.** The polar of a point *w. r. t.* an ellipse is defined exactly as in the case of a circle or a parabola, and following the method indicated there, the student can easily show that the polar of $P(x_1, y_1)$ *w. r. t.* the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\text{is}\quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

P is called the POLE of its polar.

**Example 1.** *Show that the polar of either focus of an ellipse is the corresponding directrix.*

Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

$\therefore$ the polar of the focus $(-ae,\ 0)$ *w. r. t.* this ellipse is $x.\dfrac{(-ae)}{a^2}=1$ *i.e.* $x=\dfrac{-a}{e}$

which is equation of the directrix corresponding to the focus $(-ae,\ 0)$.

**Example 2.** *Find the pole of the line* $2x-3y=b$ *w. r. t. the ellipse* $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1.$

Let $(x_1,\ y_1)$ be the pole of the line $2x-3y-6=0$. The polar of $x_1,\ y_1$, *w.r.t.*, $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ is

$$\frac{xx_1}{9}+\frac{yy_1}{4}=1 \quad \text{or} \quad \frac{xx_1}{9}+\frac{yy_1}{4}-1=0.$$

This equation and $2x-3y-6=0$ represent the same straight line.

$\therefore$ comparing co-efficients we have $\dfrac{x_1}{18}=\dfrac{y_1}{-12}=\dfrac{1}{6}.$

$\therefore$ $x_1=3,\ y_1=-2$. Hence the pole is the point $(3,\ -2)$.

**8·52.** The following properties of pole and polar follow at once :

(*i*) If the polar of P passes through Q, the polar of Q passes through P.

P and Q are called conjugate points *w. r. t.* the ellipse.

(*ii*) The pole of $lx+my+n=0$ is $\left(\dfrac{-a^2l}{n},\ \dfrac{-b^2m}{n}\right).$

(*iii*) If the pole of a line $L_1$ lies on $L_2$ then the pole of $L_2$ lies on $L_1$.

$L_1$ and $L_2$ are called **Conjugate Lines** *w.r.t.* the ellipse.

## Exercises VIII (D)

1. Find the polar of

(*i*) (2, 3) *w.r.t.* the ellipse $2x^2+y^2=1$.

(*ii*) $(-3, 4)$ *w.r.t.* the ellipse $3x^2+4y^2=5$.

2. Find the pole of the line

(*i*) $x+2y=3$ *w.r.t.* the ellipse $2x^2+3y^2=1$,

(*ii*) $3x-4y+5=0$ *w.r.t.* the ellipse $4x^2+y^2=3$.

3. Find the condition of conjugacy of two points $(x_1, y_1)$ $(x_2, y_2)$.

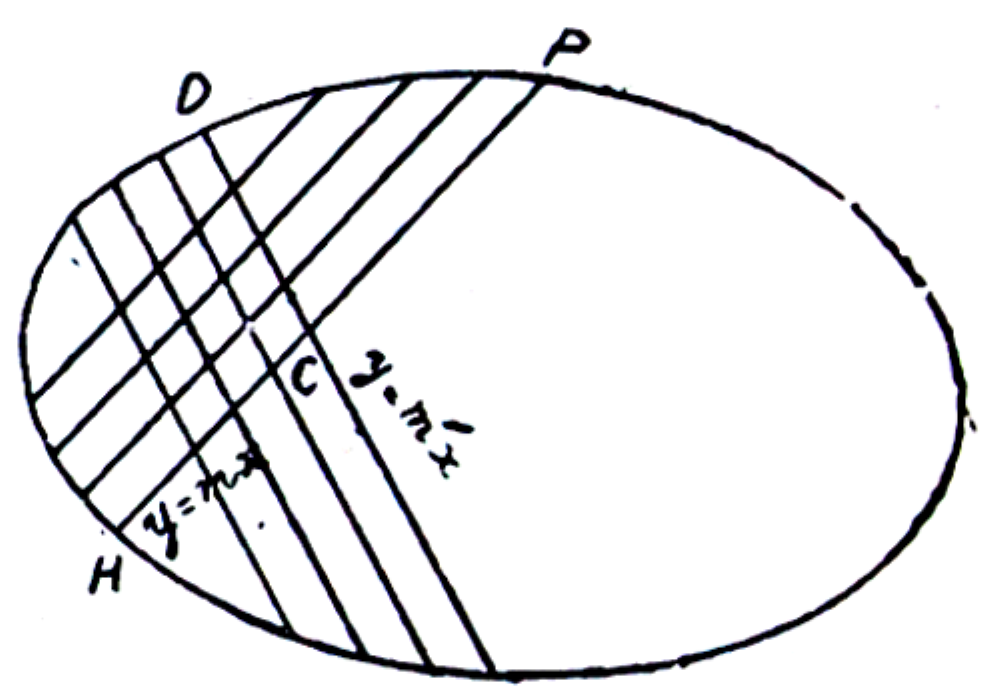
4. Show that the lines $lx+my=1$ and $l'x+m'y=1$ are conjugate lines *w.r.t.* the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if $a^2ll'+b^2mm'=1$.

5. If the polars of two points P and Q *w.r.t.* an ellipse meet in R, show that R is the pole of the line PQ.

6. Show that the condition that the pole of $lx+my=1$ *w.r.t.* the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ may lie on the ellipse $\dfrac{x^2}{4a^2}+\dfrac{y^2}{4b^2}=1$ is $a^2l^2+b^2m^2=4$.

### 8·6. Conjugate diameters.

*If the diameter $y=m'x$ bisects chords parallel to the diameter $y=mx$, then the diameter $y=mx$ bisects chords parallel to the diameter $y=m'x$.*




Let the diameter bisecting chords parallel to $y=m'x$ be $y=Kx$. Then

$$K=-\frac{b^2}{a^2m'} \qquad \text{......(1)}$$

[from Cor. Art. 8·26]

Also since $y=m'x$ is given to bisect chords parallel to $y=mx$,

$$m'=-\frac{b^2}{a^2m} \qquad \text{......(2)}$$

From (1) and (2), we have

$$K=-\frac{b^2}{a^2m'}=-\frac{b^2}{a^2}\cdot\frac{a^2m}{(-b^2)}=m.$$

Substituting for K we see that the diameter bisecting chords parallel to $y=m'x$ is $y=mx$.

**Def.** Two diameters are called conjugate diameters, if each bisects chords parallel to the other. The condition for conjugacy then is $mm'=-\dfrac{b^2}{a^2}$.

**Example 1.** *Write down the equations of the diameters which are conjugate to the diameters $x-y=0$.*

$$x+y=0,\ y=\frac{a}{b}x \text{ and } y=\frac{b}{a}x \text{ w.r.t. } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

**Example 2.** *Prove that $y+3x=0$ and $4y-x=0$ are conjugate diameters of $3x^2+4y^2=5$.*

**8·61.** *The eccentric angle of the ends of two conjugate semi-diameters differ by a right angle.*

Let P and D be the ends and $\phi$ and $\phi_1$ be their eccentric angles.

Slope of CP$=\dfrac{b\sin\phi}{a\cos\phi}$ and slope of CD $=\dfrac{b\sin\phi_1}{a\cos\phi_1}$

The condition of conjugacy *viz.* $mm'=\dfrac{-b^2}{a^2}$ therefore gives

$$\frac{b^2\sin\phi\sin\phi_1}{a^2\cos\phi\cos\phi_1}=\frac{-b^2}{a^2}$$

or $\cos\phi\cos\phi_1\sin\phi_1=0$.

giving $\cos(\phi\sim\phi_1)=0$, $\quad\therefore\ \phi\sim\phi_1=\dfrac{\pi}{2}$.

*Note.* Solution of problems on conjugate diameters are greatly simplified with the help of Art. 8·61. For if P be $(a\cos\phi,\ b\sin\phi)$, then D is $\{a\cos(90^0+\phi),\ b\sin(90^0+\phi)\}$ or $(-a\sin\phi,\ b\cos\phi)$.

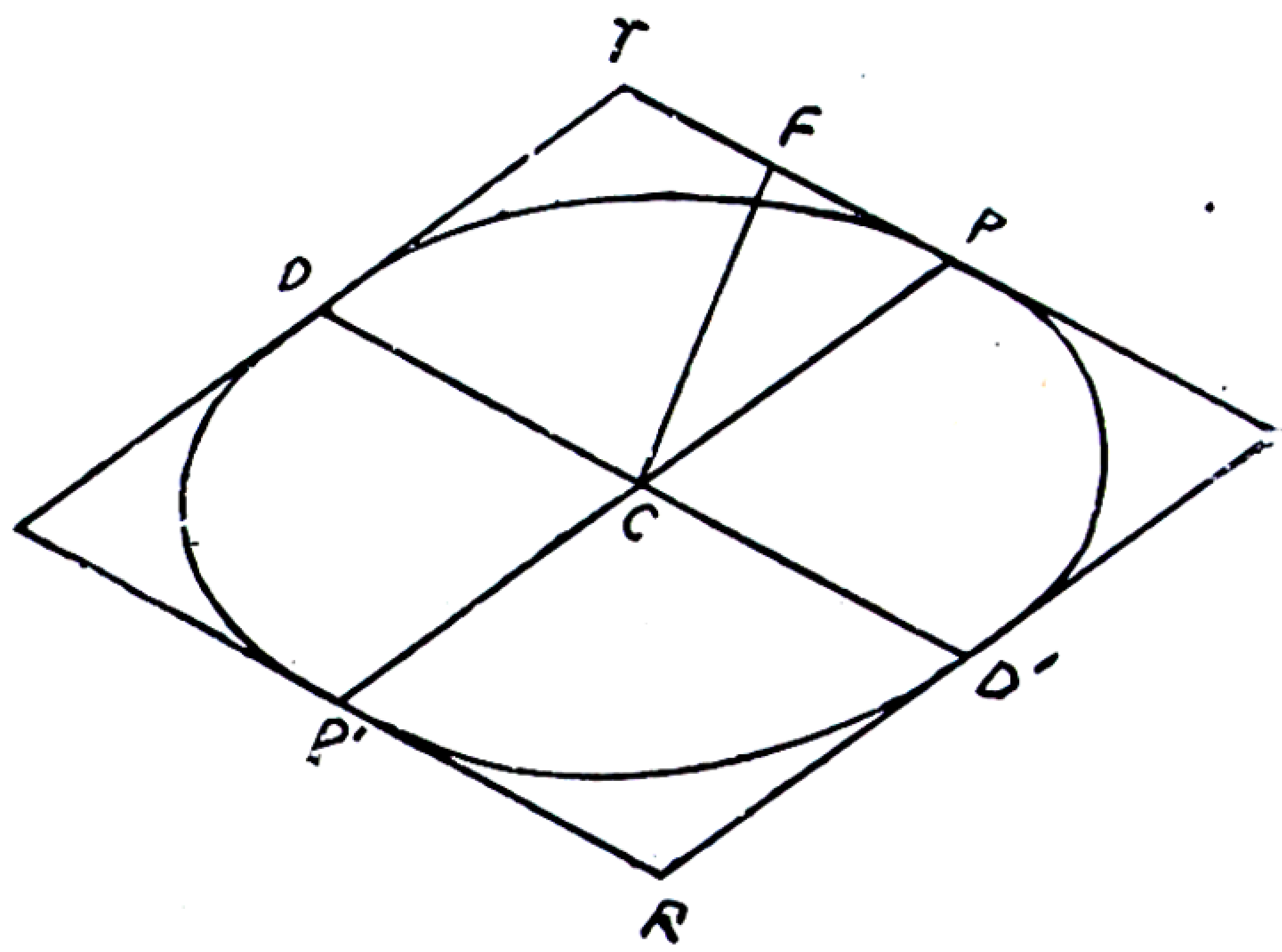
Also if P′ be the other end of the diameter through P; then P′ is $\{a\cos(180^0+\phi),\ b\sin(180^0+\phi)\}$ or $(-a\cos\phi,\ -b\sin\phi)$.

## Exercises VIII (E)

1. The sum of the squares of two conjugate semi-diameters is constant and equal to $a^2+b^2$.

2. Show that the tangent at either end of a diameter of an ellipse is parallel to the system of chords bisected by it.

3. Tangents at the ends P, P′, D, D′ of the conjugate diameters of an ellipse form a ∥ gm. RT. Semi-conjugate diameters CP, CD, form a ∥ gm. CT with the tangents at P and D. Prove that the area of the ∥ gm. RT is constant and four times that of the ∥ gm. CT.




[*Hint.* Area of ∥ gm. CT = CD × CF where CF is the length of the ⊥ from C on the tangent at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$].

4. P and D are the ends of two semi-conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the point of intersection of tangents at P and D is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

[Put down tangents at P and D and eliminate $p$]

We shall conclude this chapter with a few solved examples.

**Example 1.** *Ellipses are drawn on the same major axis. Prove that the locus of the two ends of the latera recta which are on the same side of the major axis is a parabola.*

Let $e$ be the eccentricity of an ellipse drawn on the major axis of fixed length $2a$. Then the equation of the ellipse referred to its axis as axes of co-ordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

For the upper end of the latera recta,

$z=\pm ae, \quad y=a(1-e^2); \left[\frac{b^2}{a^2}=\frac{a^2(1-e^2)}{a}=a(1-e^2),\right.$

or $\quad x^2=a^2e^2,\ ay=a^2-a^2e^2.$

Eliminating $e$ between these two equations,

$$ay=a^2-x^2 \text{ or } a^2=(a-y) \text{ which is a parabola.}$$

**Example 2.** *Find the equation of the locus of the mid-points of the portions of tangents intercepted by the axes of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ ...(1)

Let P $(a \cos \phi, b \sin \phi)$ be any point on the ellipse.

Tangent at P is $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi=1$ ...(2)

where it meets the $x$-axis $x=a \sec \phi y=0,$
where it meets the $y$-axis $x=0, \quad y=b \operatorname{cosec} \phi.$

Hence for the mid-point of the portion intercepted between the axes

$$x=\frac{a}{2} \sec \phi,\ y=\frac{b}{2} \operatorname{cosec} \phi \qquad ...(3)$$

To obtain the locus of this point, we have to eliminate $\phi$ between these equations which give

$$2 \cos \phi=\frac{a}{x},\ 2 \sin \phi=\frac{b}{y}.$$

Squaring and adding, we get

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4 \text{ as the required locus.}$$

**Example 3.** *Find the locus of poles of normal chords of an ellipse.*

Whatever '$\phi$' $\quad \frac{ax}{\cos \phi}-\frac{by}{\sin \phi}=a^2-b^2$ ...(1)

is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

If $(x_1, y_1)$ be the pole of (1), (1) must be the same as

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad ...(2)$$

Comparing co-efficients, we get

$$\frac{x_1}{a_2} \cdot \frac{\cos\phi}{a} = -\frac{y_1}{b^2} \cdot \frac{\sin\phi}{b} = -\frac{1}{a^2-b^2}. \quad \ldots(3)$$

As $\phi$ changes, the normal changes its position and the co-ordinates $(x_1, y_1)$ of the pole change. We must then eliminate $\phi$ between equations (3). From (3)

$$\cos\phi = \frac{a^3}{x_1(a^2-b^2)} \text{ and } \sin\phi = -\frac{b^3}{y_1(a^2-b^2)}.$$

Squaring and adding we have

$$\left(\frac{a^6}{x_1^2} + \frac{b^6}{y_1^2}\right) \frac{1}{(a^2-b^2)^2} = 1$$

so that $(x_1, y_1)$ lies on $\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2} = (a^2-b^2)^2$

## Revision Questions No. IV

1. Find the equation of the ellipse

(*i*) whose sum of axes $=20$, difference of axes $=4$

(*ii*) whose major axis $=12$ : minor axis $=$ distance between the foci.

(*iii*) whose latus rectum $= \dfrac{1}{2}$ ; eccentricity $= \dfrac{\sqrt{3}}{2}$.

2. (*a*) Find the equation of the ellipse

(*i*) whose focus is the origin, directrix the straight line $x\cos\alpha + y\sin\alpha = p$ and eccentricity $e$.

(*ii*) whose focus is the point $(-1, 1)$, directrix the straight line $x-y+3=0$ and eccentricity $\frac{1}{2}$. [*P. U. 1943*]

(*b*) Find the equation of the major axis and centre of ellipse in (*ii*).

3. Find the eccentricity, foci, directrices, length and equations of the latera recta of the following ellipses :

(*i*) $x^2+9y^2=81$. (*ii*) $16x^2+y^2=25$.

(*iii*) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$. [*P.U.*]

4. A point moves so that the sum of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$; prove that the locus is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$ [*P.U.* 1944]

5. Prove that the area of the triangle formed by three points on an ellipse whose eccentric angles are $\alpha$, $\beta$, $\gamma$ is $2ab \sin\frac{\beta-\gamma}{2} \sin\frac{\gamma-\alpha}{2} \sin\frac{\alpha-\beta}{2}$.

6. A straight line AB of given length has its extremities on two fixed perpendicular straight lines OA and OB. Show that the locus of any point C on AB is an ellipse whose semi-axes are OA and OB.

[**Solution**. Take the lines OA, OB for axes and let $AC=b$ and $CB=a$. Let the co-ordinates of C be $(x, y)$.

From similarity of triangles

$$\frac{x}{a}=\frac{OA}{AB} \text{ and } \frac{y}{b}=\frac{OB}{AB}.$$

Squaring and adding we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{OA^2+OB^2}{AB^2}=\frac{AB^2}{AB^2}=1].$$

7. The base of a triangle is $2a$, the sum of the remaining sides is $2b$: show that the locus of the vertex of the triangle is an ellipse.

[**Solution**. Take the base as $x$-axis and its middle point as origin. If the vertex be $(x, y)$, the co-ordinates of the ends of the base are $(-a, 0)$ and $(a, 0)$.

$\because$ the sum of the two sides is $2b$, we have

$$\sqrt{(x+a)^2+y^2}+\sqrt{(x-a)^2+y^2}=2b.$$

This on simplification gives us $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is an ellipse.

8. Find the equations of the tangents and normals to the ellipse $x^2/a^2+y^2/b^2=1$ at the ends of the latera recta.

9. Find the equations of the tangents to the ellipse $x^2/9+y^2/16=1$ at the points in which it is cut by the line $y-2=0$.

10. Prove that the equation of a tangent to the ellipse $x^2/a^2+y^2/b^2=1$ can be written as

$$x \cos \alpha+y \sin \alpha=\sqrt{a^2 \cos^2 \alpha+b^2 \sin^2 \alpha}.$$

Hence find the locus of the point of intersection of perpendicular tangents.

11. If $\theta$ be the angle between the tangents drawn from the point $(x_1, y_1)$ to the ellipse $x^2/a^2+y^2/b^2=1$ then

$$\tan \theta=\frac{2\sqrt{a^2y_1^2+b^2x_1^2-a^2b^2}}{x_1^2+y_1^2-a^2-b^2}.$$

Hence deduce the equation of the Director Circle.

12. Find a point on the ellipse $\frac{x^3}{a^-}+\frac{y^2}{b^2}=11$

(*i*) at which the tangent makes equal angles with the axes. (*P. U.*)

(*ii*) at which a tangent and a normal be drawn in order that they form, with $x$ axis as base an isosceles triangle.

13. Show that the locus of the middle points of the portion of the tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the axes of co-ordinates is the curve $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4.$

14. Find the co-ordinates of the point of intersection of normals drawn to the ellipse $x^2/a^2+y^2b^2=1$ at points whose eccentric angles are $\theta_1$ and $\phi_2$.

15. The normal at any point $P(x_1, y_1)$ meets the major axis in G, show that $AG.A'G=a^2-e^4x_1^2$ where A and A′ are the vertices of the ellipse.

[*Hint.* G is the point $(e^2x_1, 0)$, A is $(a, 0)$ and B′ $(-a, 0)$]

16. If the normal at any point P of ellipse $x^2/a^2+y^2/b^2=1$ meet the major and minor axex in G and $g$ and if CF be the

perpendicular from the centre on the normal at P, then show that

(*i*) PF.PG$=b^2$ (*ii*) PF.P$g=a^2$.

17. Find the locus of the point of intersection of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn at points the sum of whose eccentric angles is constant $(=2a)$.

[*Hint.* Let the eccentric angles of the points be $\alpha-\theta$ and $\alpha+\theta$].

18. Any ordinate NP of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in Q. Prove that

(*i*) tangents at P and Q intersect on the major axis,

(*ii*) normals at P and Q intersect on the circle $x^2+y^2=(a+b)^2$.

[*Hint.* If the point P be $(a \cos\theta, b \sin\theta)$, the point Q is $(a \cos\theta, a \sin\theta)$. Solve simultaneously the equations of the tangents at P and Q to the ellipse and the circle respectively).

19. If $fx+gy+h=0$ is the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point whose normal passes through the point $(x_1, y_1)$, show that $(a^2-b^2)fg+x_1gh-y_1h_1=0$. (*P.U.*)

[**Solution**. Let $fx+gy+h=0$ be a tangent at the point $(\alpha, \beta)$.

Then (1) is identical with $\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}-1=0$

$$\therefore \quad \frac{\alpha}{a^2f}=\frac{\beta}{b^2g}=-\frac{1}{h} \quad \text{or} \quad \alpha=-\frac{a^2f}{h},\ \beta=-\frac{b^2g}{h}.$$

$\therefore$ the point of contact is $\left(\frac{-a^2f}{h}, \frac{-b^2g}{h}\right)$.

Normal at this point is $$\frac{x+\frac{a^2f}{h}}{-\frac{af}{a^2h}}=\frac{y+\frac{b^2g}{b^2k}}{-\frac{b^2g}{b^2h}}$$

or $xhg-yhf+(a^2-b^2)fg=0$.

$\because$ This passes through $(x_1, y_1)$ we have

$x_1hg-y_1hf+(a^2-b^2)fg=0$]

20. Tangent at any point P of an ellipse meets the axes in Q and R. The rectangle CQTR is completed. Show that the locus of T is $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$.

[*Hint.* As in Q. 13 the points Q and R are $\left(\dfrac{a}{\cos\phi}, 0\right)$ and $\left(\theta, \dfrac{b}{\sin\theta}\right)$ and the middle point S of QR is $\left(\dfrac{a}{2\cos\theta}, \dfrac{b}{\sin\theta}\right)$. Since S is the middle point of CT

$\therefore$ T is $\left(\dfrac{a}{\cos\theta}, \dfrac{b}{\sin\theta}\right)$].

21. Show that the lines $2x-y=0$ and $x+3y=0$ are conjugate diameters of the ellipse $2x^2+3y^2=4$.

22. Find a common tangent to the ellipses

$$(i)\quad \frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1,$$

$$(ii)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

[*Hint.* (*i*) Tangent to the first ellipse is $y=mx+\sqrt{(a^2+b^2)m^2+b^2}$ · Find the value of $m$ so that it may be tangent to the second.]

23. Show that the locus of the poles of tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *w.r.t.* the circle $x^2+y^2=a^2$ is the ellipse $a^2x^2+b^2y^2=a^4$.

[**Solution.** Let the pole be $(x_1, y_1)$ Then its polar *w. r. t.* $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$. If this is a tangent to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ we have $a^2x_1^2+b^2y_1^2=a^2$,

$\therefore$ locus of $(x_1\ y_1)$ is $a^2x^2+b^2y^2=a^4$].

24. Prove that the locus of a point which moves so that the perpendicular distance of the centre from its polar *w. r. t.* the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is constant and equal to $e$ is $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}=\dfrac{1}{c^2}$.

# CHAPTER IX

## GRAPHIC SOLUTION OF QUADRATIC EQUATIONS

**9·1. Graph of an equation.** It has been shown how a specified locus or curve may be represented by an equation; the converse problem is to represent a given equation by its graph.

If we know that the locus of a given equation is a straight line, its graph can easily be drawn; it is only necessary to find any two points on it, plot them and draw a straight line through them. Likewise, if we know that the locus is a circle and can find its centre and radius, the graph can be drawn with the help of a pair of compasses.

When the form of the curve is not known, the most straightforward way of obtaining its graph is to calculate the co-ordinates of a number of points, plot the points and draw the curve through them. The main thing to observe is that, the point must be close enough together to enable us to be sure that we have found the general trend of the curve.

**9·11.** *Rule to draw the graph of a given equation.*

*First step.* Solve the given equation for one of the variables in terms of the other. Always choose the simplest solution.

*Second step.* By the formula find the values of the variable for which the equation has been solved by assigning real values to the other variable.

*Third step.* Choose a suitable scale of length and plot the points whose co-ordinates have been obtained above.

*Fourth step.* Draw a continuous curve through the points.

*Notes.*—1. The above method of drawing a graph is from its nature an approximate method. But, more the number of points plotted and nearer they are to each other, the more accurate is the graph of the curve.

2. In practice the unit of length should be determined by the size of the graph paper compared with the greatest length to be laid off upon it.

**Example 1.** *Draw the graph of the curve* $x^2+y^2-6x+5=0$. *Read the points of its intersection with the x-axis.*

**First Method**

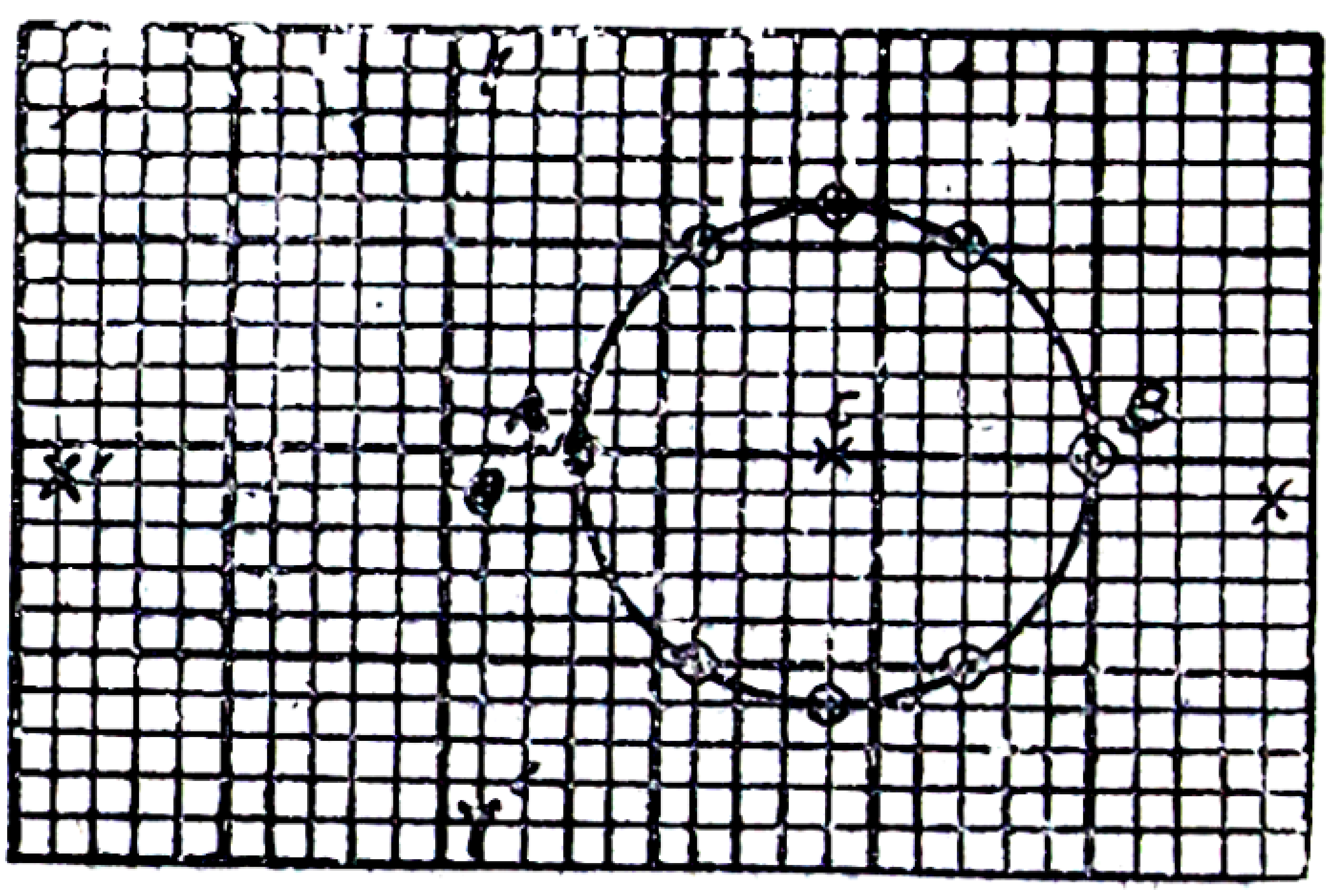
*First step.* Solving for $y$,

$$y=\pm\sqrt{6x-5-x^2}.$$

*Second step.* By assigning to $x$ values differing by unity and finding the corresponding values of $y$, we obtain the table of values given below :

| $x$ | $y$ | $x$ | $y$ |
|---|---|---|---|
| 0 | Imag. | | |
| 1 | 0 | $-1$ | Imag. |
| 2 | $\pm1{\cdot}7$ | $-2$ | Imag. |
| 3 | $\pm\ {\cdot}2$ | etc. | etc. |
| 4 | $\pm1{\cdot}7$ | | |
| 5 | 0 | | |
| 6 | Imag. | | |
| etc. | etc. | | |

1 unit=3 times the side of a small square.



*Third step.* Plot the points (1, 0), (2, 1·7), (2, −1·7) etc. etc.

*Fourth step.* Draw a continuous curve through these points. This gives the required graph as shown in the figure.

**Second Method**

The equation $x^2+y^2-6x+5=0$ represents a circle whose centre is (3, 0) and whose radius is 2. Plot the point (3, 0) with this as centre and radius=six times the side of small square $(2\times3=6)$ draw a circle. This gives the required graph.

This circle meets the $x$-axis in the points A and B where OA $=\frac{1}{3}$. $3=1$, and OB $=\frac{1}{3}$. $15=5$. $\therefore$ The points of intersection with the $x$-axis are $(1, 0)$, $(5, 0)$.

This can be verified by solving the equation of the circle simultaneously with $y=0$, the equation of the $x$-axis.

**9·12. Graphic Solution of Equations.** Graphs of curves are often useful for obtaining approximations to the roots of the equations. The following examples illustrate the method.

**Example 2.** *Find graphically the roots of the equation $2x^2+4x-3=0$ correct to first figure after the decimal point.* [*P.U. 1942*]

The roots of the given equation are the abscissæ of the points of intersection of the circle $2(x^2+y^2)+4x-3=0$, with $y=0$ *i.e.*, $x$-axis.

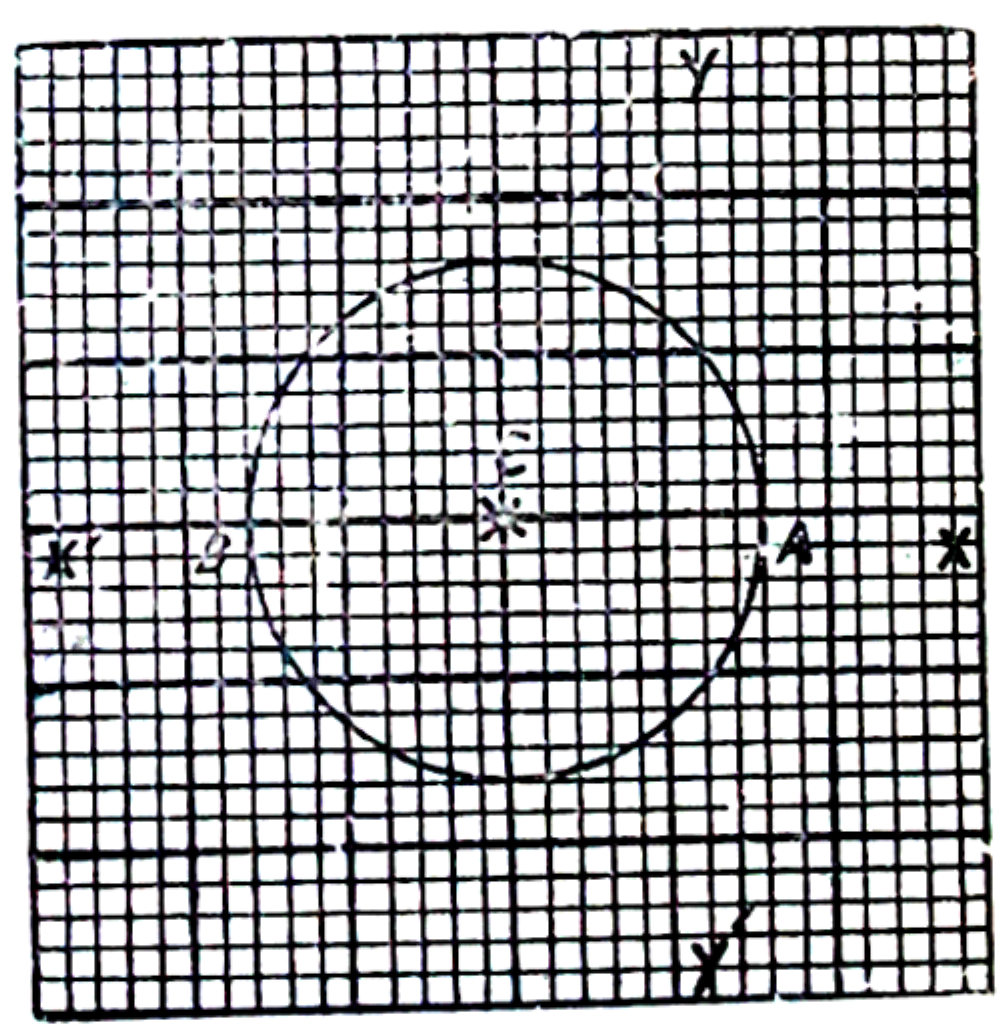
The equation of the circle can be written as

$$x^2+y^2+2x-\tfrac{3}{2}=0$$

$\therefore$ its centre is $(-1, 0)$ and radius $=\sqrt{\frac{5}{2}}=\sqrt{\frac{10}{4}}$.

$$=\tfrac{1}{2}\times 3{\cdot}2=1{\cdot}6.$$

One unit $=5$ times the side of a small square.




Plot the point C $(-1, 0)$. With C as centre and radius equal to 8 times the side of a small square $(1{\cdot}6\times 5=8)$, draw a circle.

If this circle meets the $x$-axis in the points A and B. Then OA and OB are the roots of the given equation.

$$\text{OA}=\tfrac{1}{5}.\ 3={\cdot}6$$

$$\text{OM}=\tfrac{1}{5}\ (-13)=-2{\cdot}6$$

$\therefore$ the roots are $\cdot 6$ and $-2{\cdot}6$.

# CHAPTER X

## THE HYPERBOLA

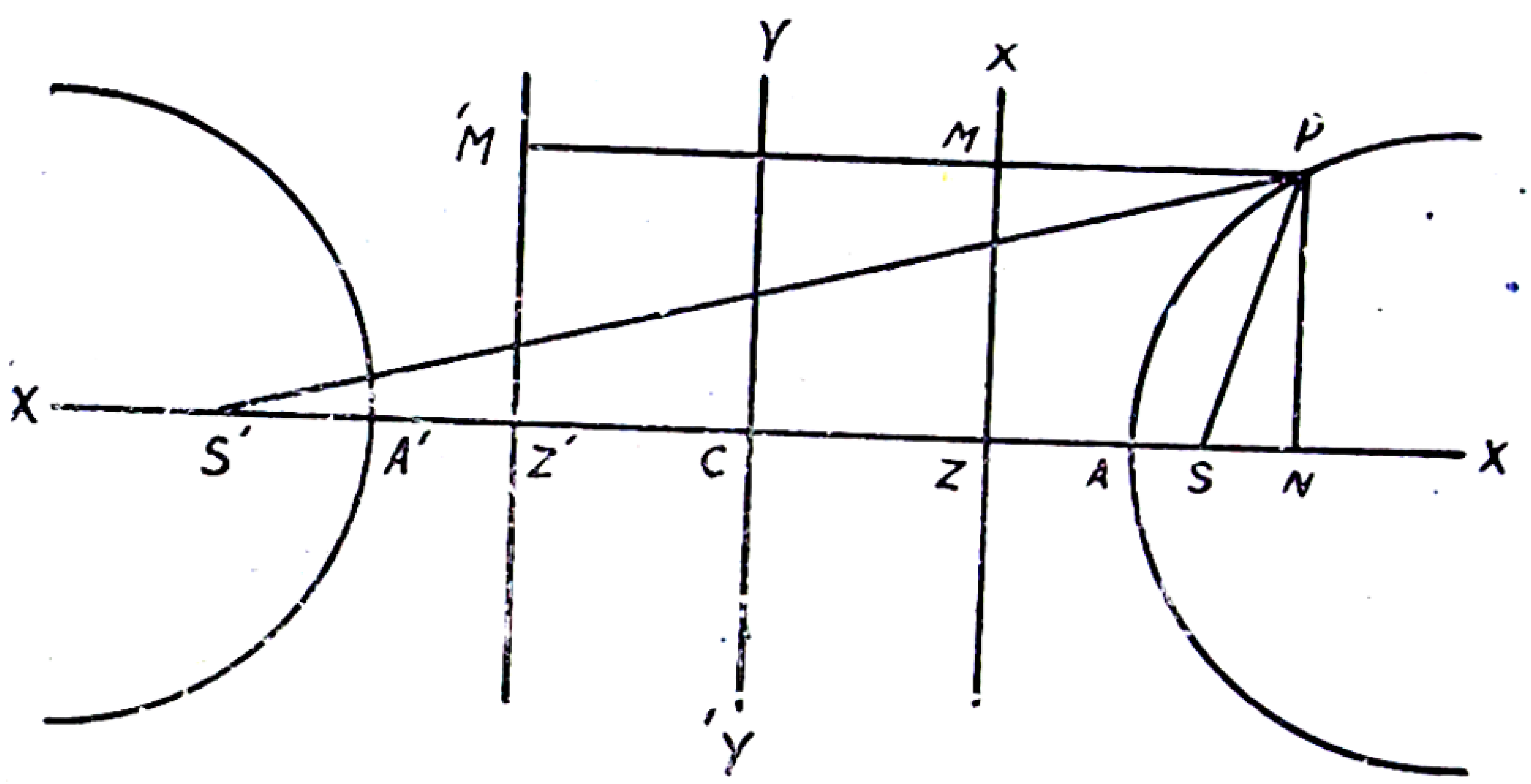
**10. Def.** *A hyperbola is the locus of a point which moves in such a way that its distance from a* **fixed point** *bears a constant ratio (more than one) to its distance from a* **fixed straight line**.

The fixed point is called **focus**, the fixed straight line the **directrix** and the constant ratio is called the **eccentricity**. The latter is denoted by $e$.

**10·1.** *Equation of a hyperbola in simplified form.*

Let S be the focus and ZM the directrix. Draw SZ perpendicular to the directrix and divide it internally and externally at A and A′ in the ratio $e : 1$ so that A and A′ are points on the locus.

Let C be the middle point of AA′ and let $A'A = 2a$.




Then we have $A'S = eA'Z$ (*i*) $AS = eZA$ (*ii*)

Adding (*i*) and (*ii*)

$A'S + AS = e(A'Z + ZA) = eA'A.$

$\therefore\ 2CS = 2ae.$ $\qquad \therefore\ CS = ae.$

Subtracting (*ii*) from (*i*) we get

$A'S - AS = e(A'Z - ZA)$

$AA' = e\,[A'C + CZ - (CA - CZ)]$ or $2a = 2e.\,CZ.$

$\therefore\quad CZ = \dfrac{a}{e}.$

Take C as origin and the $x$-axis along CZ and let $(x, y)$ be the co-ordinates of any point P on the locus, NP the ordinates of P and PM the perpendicular to the directrix.

Then $SP^2=e^2PM^2=e^2NZ^2$.

$\therefore\quad SN^2+NP^2=e^2(CN-CZ)^2$

or $\qquad (x-ae)^2+y^2=e^2\left(x-\dfrac{a}{e}\right)^2=(ex-a)^2$

or $x^2(e^2-1)-y^2=a^2(e^2-1)$

and dividing by $a^2(e^2-1)$ this gives

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1 \qquad \ldots(1)$$

Since $e$ is greater than unity $a^2(e^2-1)$ is a positive number and, if we take $b^2$ for $a^2(e^2-1)$ the equation becomes

$$\mathbf{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1} \qquad \ldots(2)$$

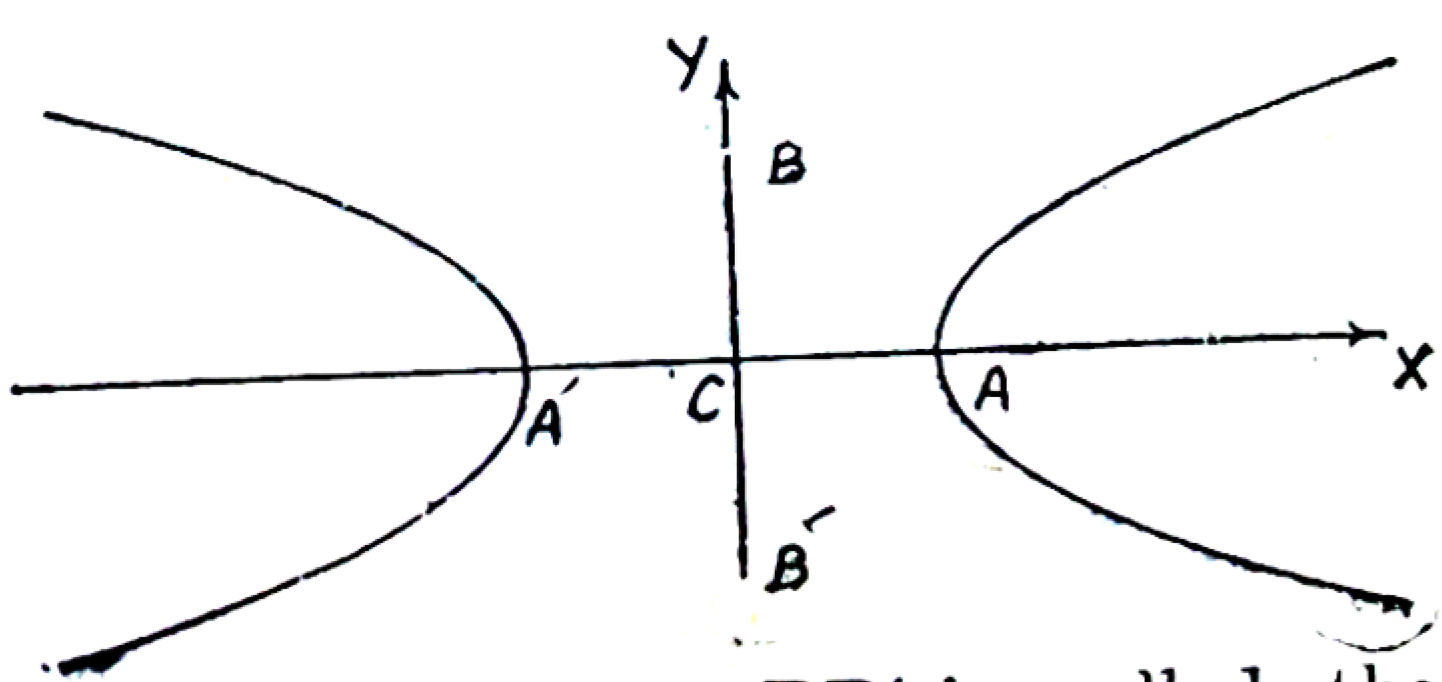
and this is the standard form of equation of a hyperbola.

**10·11.** Putting $y=0$ in the equation $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, we get

$$x=\pm a$$

so that the $x$-axis meets the hyperbola in points A $(a, 0)$ and A$'(-a, 0)$.

The points A, A$'$ are called the **vertices** and AA$'$ the **transverse** axis of the hyperbola.




Now putting $x=0$ in the same equation, we get $y^2=-b^2$

$y$-axis meets the hyperbola in imaginary points. If however, we take on the $y$-axis two points B and B$'$ such that CB=CB$'=b$, then BB$'$ is called the **conjugate** axis.

C is called the **centre** of hyperbola.

**Latus Rectum.** The chord of the hyperbola passing through the focus and $\perp$ to AA$'$ is called a Latus rectum.

**Rectangular Hyperbola.** When $a=b$, the hyperbola is called a rectangular hyperbola. The equation of a rectangular hyperbola, thus, is $x^2-y^2=a^2$

## 10·12. Shape of a hyperbola.

As we have already seen the $x$-axis meets the hyperbola in two points A $(a, 0)$ and A$'(-a, 0)$. The $y$-axis does not meet the curve in the real points.

Now writing the equation of the curve in the form

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2} \qquad (i)$$

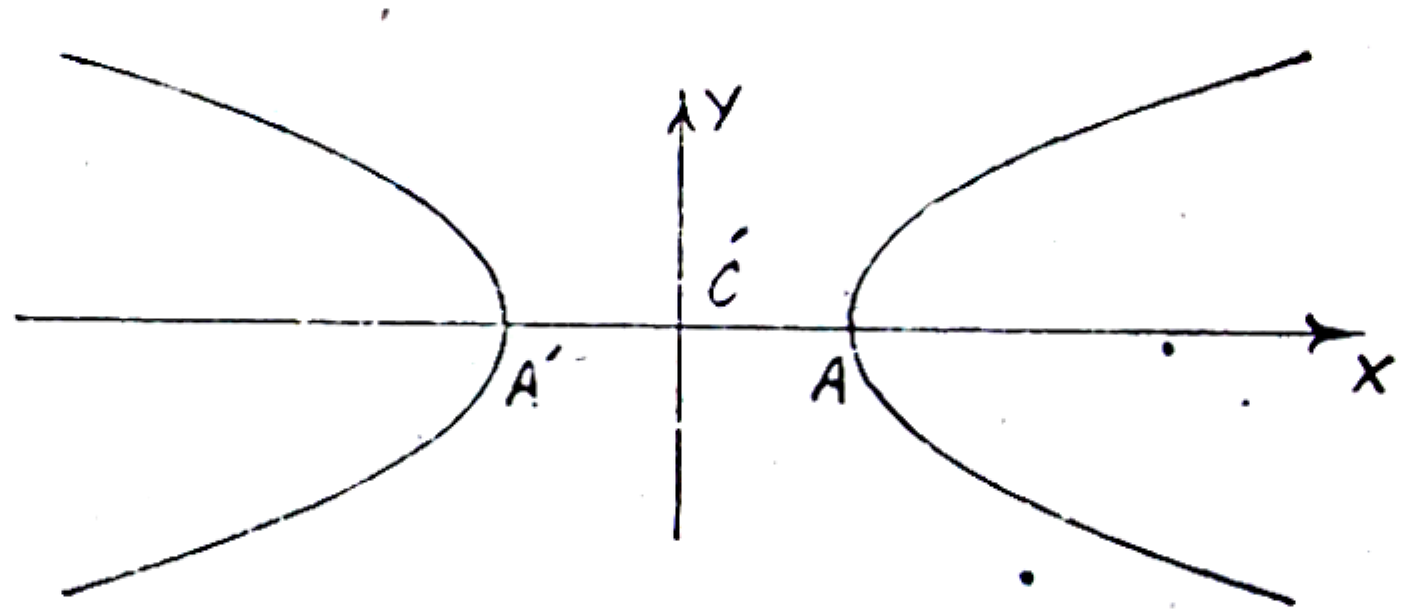
we find that for values of $x$ lying between $-a$ and $a$, $y$ is imaginary. Thus no part of the curve lies between the lines $x=\pm a$.

For values of $x$, not lying between $-a$ and $a$, there are two equal and opposite values of $x$ $\therefore$ the curve is *symmetrical about the x-axis.*

Again writing the equation in the form $x=\pm\frac{a}{b}\sqrt{b^2+y^2}$,

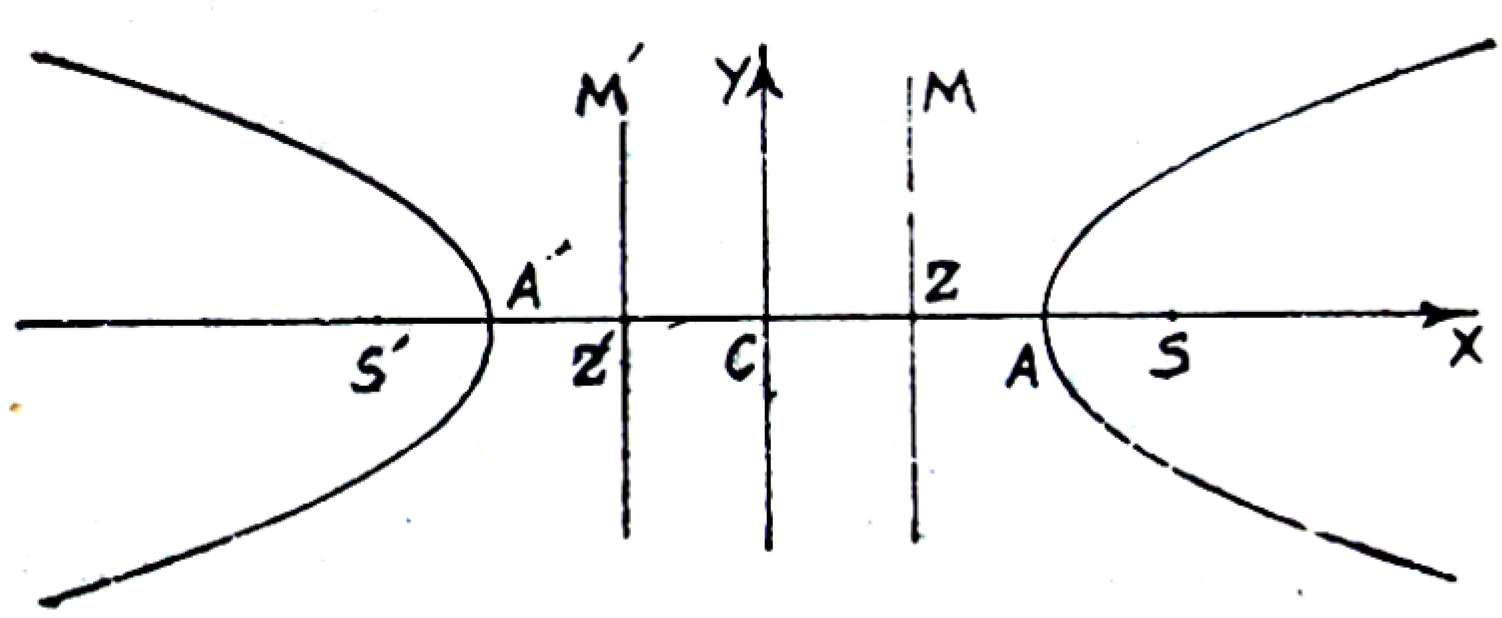
it is clear that for every value of $y$, there are two equal and opposite values of $x$, $\therefore$ the curve is *symmetrical about the y-axis.*

It is also clear from equation $(i)$ that $y$ increases with $x$ and there is no limit to this increase. The shape of the curve, therefore, is as shown.




## 10·13. A hyperbola has two foci and two directrices.

On the line CA$'$, take the points Z$'$ and S$'$ such that CZ$'$=CZ $=a/e$ and CS$'$=CS $=ae$.

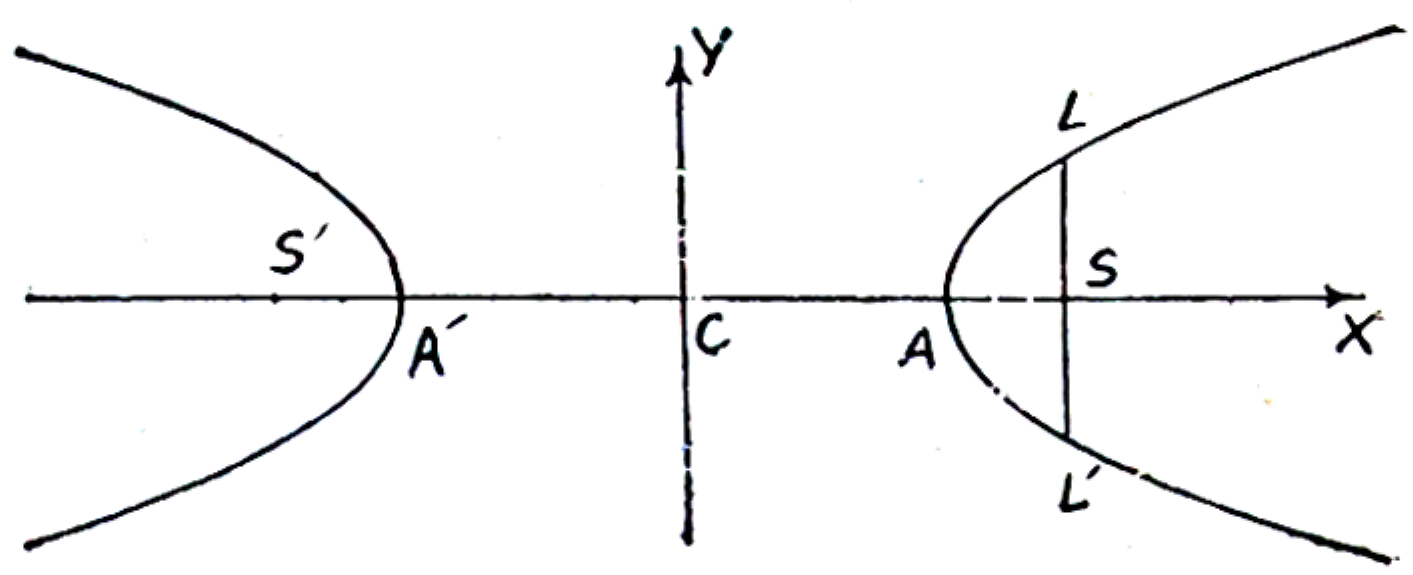



Through Z$'$ draw M$'$Z$'$ perpendicular to AA$'$. The hyperbola, being symmetri-

cal about the $y$-axis, S′ will be the second focus and Z′M′ the second directrix.

The co-ordinates of the two foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(ae, 0)$ and $(-ae, 0)$ and the equations of the corresponding directrices are $x=\frac{a}{e}$ and $x=\frac{-a}{e}$.

**10·14.** *To find the length of the latus rectum of the hyperbola.*




$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Let LSL′ be the latus rectum. The co-ordinates of the point L are (CS, SL) *i.e.* $(ae, \text{SL})$.

Since it lies on the hyperbola, we have

$$\frac{a^2e^2}{a^2}+\frac{\text{S}'\text{L}^2}{b^2}=1$$

$$\therefore \quad \frac{\text{S}'\text{L}^2}{b^2}=1-e^2$$

or $\text{S}'\text{L}^2=b^2(l-e^2)=b^2\times\frac{b^2}{a^2}=\frac{b^4}{a^2}$

$$\therefore \text{S}'\text{L}=\frac{b^2}{a}$$

Hence the length of the latus rectum is $\frac{2b^2}{a}$.

## Exercises X (A)

Find the equation to the hyperbola, referred to its axes as axes of co-ordinates :

1. Whose conjugate axis is 7 and which passes through the point (3, 2).

2. The distance between whose foci is 8 and whose eccentricity is $\sqrt{2}$.

3. Whose conjugate axis is 2 and the distance between whose foci is $2\sqrt{5}$.

4. In the hyperbola $4x^2-9y^2=36$, find the axes, the co-ordinates of the foci, the eccentricity and the latus rectum.

5. Find the eccentricity and the co-ordinates of the foci of the hyperbola

$$3x^2-y^2=4.$$

6. Trace the hyperbola $4y^2-25x^2=100$. Find the co-ordinates of foci, the equations of the directrices, the eccentricity and the latus-rectum.

**10·2. Equation of the tangent at a point.** To find the equation of the tangent at the point P $(x_1, y_1)$ of the hyperbola.

Let Q $(x_2, y_2)$ be another point on the hyperbola.

The equation of the line passing through P $(x_1, y_1)$ and Q $(x_2, y_2)$ is

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2} \qquad \ldots(1)$$

Since P and Q lie on the hyperbola, we have

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \qquad \ldots(2)$$

and $$\frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}=1 \qquad \ldots(3)$$

$$\therefore \quad \frac{x_1^2-x_2^2}{a^2}-\frac{y_1^2-y_2^2}{b^2}=0$$

or $$\frac{y_1-y_2}{x_1-x_2}=\frac{b^2}{a^2}\cdot\frac{x_1+x_2}{y_1+y_2}$$

Substituting this value of $\frac{y_1-y_2}{x_1-x_2}$ in (1), equation of the chord PQ is

$$\frac{y-y_1}{x-x_1}=\frac{b^2}{a^2}\cdot\frac{x_1+x_2}{y_1+y_2} \qquad \ldots(4)$$

Now the limiting position of this chord when Q tends to P is the tangent at P.

$\therefore$ Putting $x_2=x_1$, $y_2=y_1$, in (4), the equation of tangent at $(x_1, y_1)$ is

$$\frac{y-y_1}{x-x_1}=\frac{b^2}{a^2}\cdot\frac{x_1}{y_1}$$

or $$\frac{y_1}{b^2}(y-y_1)=\frac{x_1}{a^2}(x-x_1)$$

or $$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{x_1^2}{a^2}$$

or $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

$$=1$$

$\therefore$ $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

is the required tangent.

**10·21. Equation of the normal.** To find the equation of the normal at the point P $(x_1, y_1)$ of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

The slope of the tangent

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

at the point $(x_1, y_1)$ is $\dfrac{b^2x_1}{a^2y_1}$.

$\therefore$ the slope of the normal at $(x_1, y_1)$ is $-\dfrac{a^2y_1}{b^2x_1}$.

Hence the required equation of the normal is

$$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1).$$

or $$\frac{x-x_1}{\dfrac{x_1}{a^2}}=\frac{y-y_1}{\dfrac{-y_1}{b^2}}$$

which can also be written as

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2.$$

**10·22. Intersection of a line and a hyperbola.** To find the points of intersection of the line $y=mx+c$ with the hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

The equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \ldots (1)$$

and that of the line is

$$y=mx+c \quad \ldots (2)$$

Solving (1) and (2) simultaneously, we have

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

or $b^2x^2-a^2(m^2x^2+2mcx+c^2)=a^2b^2$
or $x^2(a^2m^2-b^2)+2a^2cmx-a^2(c^2+b^2)=0$

which is a quadratic in $x$ and gives two values of $x$.

Substituting these values of $x$ in (2), we get the corresponding values of $y$.

*Note.* **As must be clear by now, most of the results for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are obtained by methods exactly similar to those employed in case of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.**

We will, therefore, leave the proofs of the following articles as exercises to the students.

**10·23. Length of the intercept.** Find the length of the intercept made by the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, on the line $y=mx+c$ and deduce the condition that the line may touch the hyperbola.

**11·24. Conditon of Tangency.** To find the condition that the line $y=mx+c$ may touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

**Cor.** Show that $y=mx+\sqrt{a^2m^2-b^2}$ is a angent to the hyperbola for all values of $m$.

## Exercises X (B)

1. Find the equations to the tangents $4x^2-3y^2=24$ and normals to the hyperbola at points where it meets the line $y=2$

2. (a) Find the points common to the hyperbola $2x^2-3y^2=20$ and the line $x=2y$.

(b) Find the length of the straight line intercepted by the hyperbola.

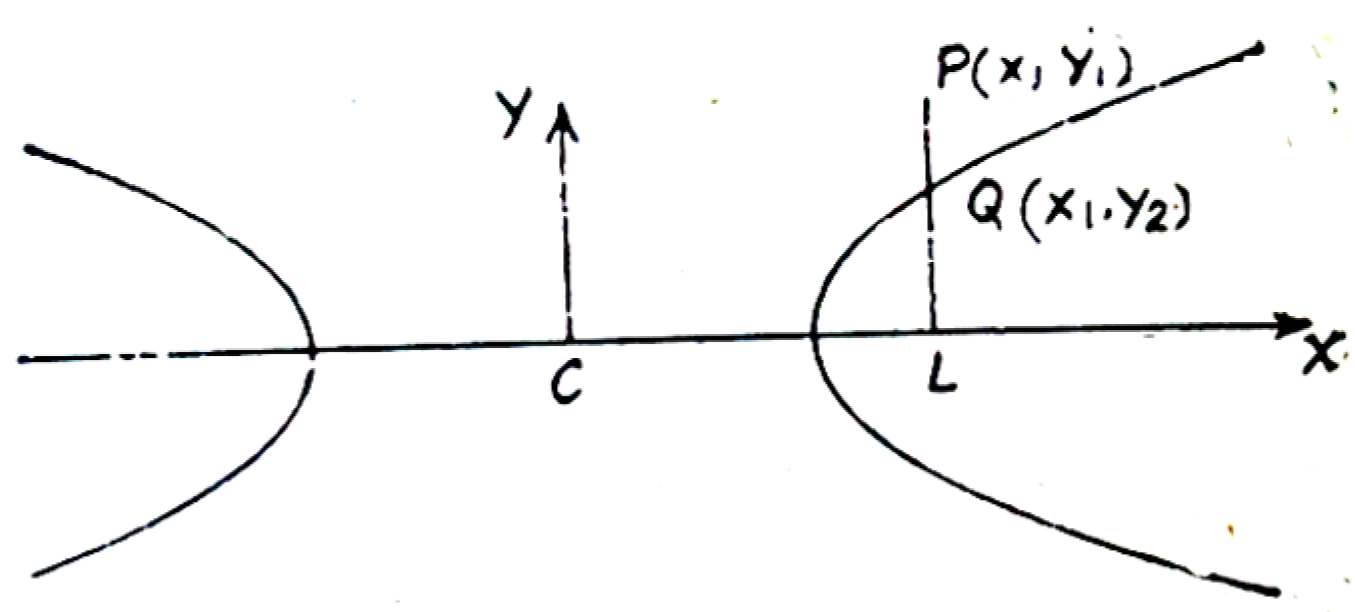
3. Find the equations of the tangents to the hyperbola $4x^2-3y^2=5$ which are parallel to the line $y=3x+7$.

4. Find the tangents to the hyperbola $4x^2-3y^2=1$ which are perpendicular to the line $3x-4y+5=0$.

5. Show that the line $x\cos\alpha+y\sin\alpha=p$ will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$.

6. Show that the line $lx+my=n$ will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2l^2-b^2m^2=n^2$.

**10·3. Position of a point relative to the hyperbola.** To show that a point $(x_1, y_1)$ will be outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}<1$, $=1$ or $>1$.




From the given point P $(x_1, y_1)$ draw PL $\perp$ on $x$-axis meeting the hyperbola in Q $(x_1, y_2)$.

Now Q $(x_1, y_2)$ lies on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\therefore \quad \frac{x_1^2}{a^2}-\frac{y_2^2}{b^2}=1$$

$$\text{or} \quad y_1^2=b^2\left(\frac{x_1^2}{a^2}-1\right)$$

Now P will lie outside the hyperbola if 2P is numerically greater than 2Q *i.e.*,

$$2P^2 > 2Q^2$$

or $$y_1^2 > b^2\left(\frac{x_1^2}{a^2} - 1\right)$$

or $$\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2} + 1 > 0$$

or $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0.$$

P will lie on the hyperbola if

$$2P^2 = 2Q^2$$

or $$y_1^2 = b^2\left(\frac{x_1^2}{a^2} - 1\right)$$

or $$\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2} + 1 = 0$$

or $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0$$

and P will lie inside if

$$2P^2 < 2Q^2$$

or $$y_1^2 < b^2\left(\frac{x_1^2}{a^2} - 1\right)$$

or $$\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2} + 1 < 0$$

or $$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$$

$$\frac{y_2^2}{b^2} = \frac{x_1}{a^2} - 1$$

$$y_2^2 = b^2\left(\frac{x_1^2}{a^2} - 1\right)$$

Now P.

$$2P > 2Q$$

$$y_1^2 > b^2\left(\frac{x_1^2}{a^2} - 1\right)$$

$$\frac{y_1^2}{b^2}-\frac{x^2}{a^2}+1>0.$$

**10·31. Tangents from a point.** To prove that from any point two tangents can be drawn to a hyperbola, and that they are real, coincident, or imaginary according as the point is outside, on or inside the hyperbola.

[**Hint**. Proceed as in the case of the corresponding article on ellipse.]

**Example 1.** *Find the locus of the point of intersection of perpendicular tangents to the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The line $y=mx+\sqrt{a^2m^2-b^2}$ ...(1)

is a tangent to the hyperbola for all values of $m$. Hence the equation of the perpendicular tangent is

$$y=-\frac{1}{m}x+\sqrt{\frac{a^2}{m^2}-b^2}$$

or $my+x=\sqrt{a^2-b^2m^2}$ ...(2)

To obtain the equation of the locus of their point of intersection, we have to eliminate $m$ between (1) and (2).

Squaring and adding we get

$$y^2+m^2x^2-2mxy=a^2m^2-b^2$$
$$m^2y^2+x^2+2mxy=a^2-b^2m^2$$
$$(1+m^2)(x^2+y^2)=(1+m^2)(a^2-b^2)$$
$$\therefore \quad x^2+y^2=a^2-b^2.$$

is the equation of the required locus which is evidently a circle having its centre as the origin.

This is called the **Director circle**.

**Example 2.** *The locus of the feet of the perpendiculars from the foci on any tangent is the auxiliary circle.*

The equation of any tangent is

$$y=mx+\sqrt{a^2m^2-b^2}. \quad ...(1)$$

This line through $(ae, 0)$ perpendicular to (1) is

$$y-0=\frac{-1}{m}(x-ae). \quad ..(2)$$

To obtain the locus we eliminate $m$ between the two equations.

$$y^2+m^2x^2-2mxy=a^2m^2-b^2$$
$$m^2y^2+x^2+2mxy=a^2e^2$$
$$=a^2+b^2$$

Adding the two equations we get $x^2+y^2=a^2$ as the locus required which is obviously a circle.

This circle is called **auxiliary circle.**

3. Find the equations of the tangents to the hyperbola $3x^2-4y^2=12$, from the point (2, 3).

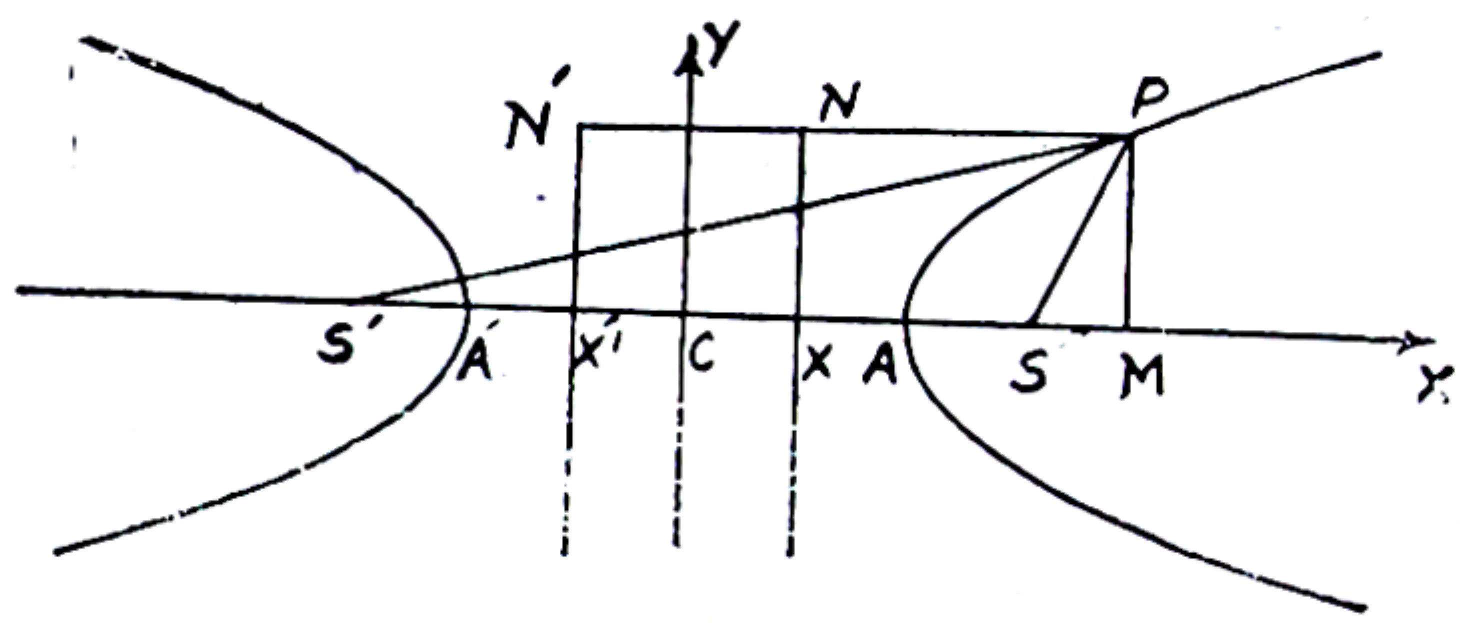
## Geometrical Properties

**10·4.** *Show that the difference of the focal distances of any point on the hyperbola is constant and equal to the transverse axis.*

Let P $(x_1, y_1)$ be any point on the hyperbola with S and S′ as its foci.

Join SP and S′P

Draw PNN′ ⊥ to the directrices meeting the same in M and M′.




Draw PM ⊥ on the $x$-axis.

Now $SP=e.\ PN=e.MX=e(CM-CX)$

$$=e\left(x_1-\frac{a}{e}\right)=ex_1-a$$

$$SP'=e.PN'=e.MX'=e(CM+CX')$$

$$=e\ (x_1+\frac{a}{e})=ex_1+a$$

$\therefore\quad S'P-SP=ex_1+a-(ex_1-a)=2a$

**10·41.** [*Locus of the middle points of a system of parallel chords of a hyperbola.*

The equation of the chord joining the points $(x_1, y_1)$, $(x_2, y_2)$ on the hyperbola is

$$\frac{y-y_1}{x-x_1}=\frac{b^2}{a^2}\cdot\frac{x_1+x_2}{y_1+y_2}$$ (by Art. 7)

$\therefore$ the slope of the chords

$$m=\frac{b^2}{a^2}\left(\frac{x_1+x_2}{y_1+y_2}\right)$$

If $(h, k)$ be the middle point of the chord, then

$$h=\frac{x_1+x_2}{2}, \qquad k=\frac{y_1+y_2}{2}$$

$$\therefore \qquad m=\frac{b^2}{a^2}\cdot\frac{2h}{2k}$$

$$\text{or } k=\frac{b_2}{a^2m}h$$

$\therefore$ the required locus is the equation

$$y=\frac{b^2}{a^2m}x$$

which is a straight line passing through the centre. This is called a **Diameter**.

$$\text{If} \quad y=\frac{b^2}{a^2m}x \text{ be written as } y=m'x.$$

$$\text{then } m'=\frac{b^2}{a^2m}$$

$$\text{or } mm'=\frac{b^2}{a^2}.$$

**Example.** *Find the equation of the chord of the hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$ which is bisected at the point $(2, 3)$.*

If the slope of the chord be $m$ its equation is

$$y-3=m(x-2)$$

$$\text{but} \qquad m=\frac{b^2}{a^2}\cdot\frac{h}{k}=\frac{3}{4}\cdot\frac{2}{3}=\tfrac{1}{2}$$

or the equation is $x-2y+4=0$.

## Exercises X (C)

1. The tangent at any point P of a hyperbola meets the transverse axis in T and the conjugate axis in $t$. N and $n$ are the feet of the perpendiculars from P on the axis. Prove that

$$\text{CN. CT}=a^2 \text{ and}$$
$$\text{CN. C}t=-b^2$$

2. Prove that the tangent and the normal at any point of a hyperbola bisect respectively the internal and the external angle between the focal distances of the point.

3. The normal at any point P of a hyperbola meets the transverse axis in G and conjugate axis in $g$. F is the foot of the perpendicular from the centre C on the normal. Prove that

$$PF.\ PG=b^2$$
$$PF.\ Pg=a^2$$

**10·5. Chord of Contact.** As was proved in case of the parabola and ellipse, it can be shown that the equation of the chord of contact of tangents from P $(x_1, y_1)$ to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

**10·51. Pole and Polar.** The polar of a point *w.r.t.* a hyperbola is defined exactly as in the case of a parabola or an ellipse. The student can easily show that the polar of $P(x_1, y_1)$ *w. r. t.* the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is}$$

$$\frac{x\,x_1}{a^2}-\frac{yy_1}{b^2}=1.$$

P is called the pole of the polar.

**Example 1.** *Show that the polar of either focus of the hyperbola is the corresponding directrix.*

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$\therefore$ the polar of the focus $(-ae, 0)$ *w. r. t.* the hyperbola is

$$\frac{x(-ae)}{a^2}=1 \text{ or } x=\frac{-a}{e}$$

which is the equation of the directrix corresponding to the focus.

**Example 2.** *Find the pole of the line $2x-3y=6$ w.r.t. the hyperbola*

$$\frac{x^2}{9}-\frac{y^2}{4}=1.$$

Let $(x_1, y_1)$ be the pole of the line $2x-3y-6=0$.
The polar of $(x_1, y_1)$ is

$$\frac{xx_1}{9}-\frac{yy_1}{3}=1.$$

$\therefore$ The two equations represent the same straight line.

$\therefore$ Comparing co-efficients we have,

$$\frac{x_1}{18}=\frac{y_1}{12}=\frac{1}{6}.$$

$\therefore$ $x_1=3$ and $y_1=2$.

Hence the pole is the point (3, 2).

**10·52.** The following properties of pole and polar follow at once :

(*i*) If the polar of P passes through Q the polar of Q passes through P.

P and Q are called Conjugate points.

(*ii*) The pole of $lx+my+n=0$ is $\left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$

(*iii*) If the pole of $l_1$ lies on $l_2$ then the pole of $l_2$ lies on $l_1$.

$l_1$ and $l_2$ are called Conjugate lines.

## Exercises X (D)

1. Find the polar of

(*i*) the point (2. 3) *w.r.t.* the hyperbola $2x^2-y^2=1$.

(*ii*) the point $(-3, 4)$ *w.r.t.* the hyperbola $3x^2-4y^2=5$.

2. Find the pole of the line

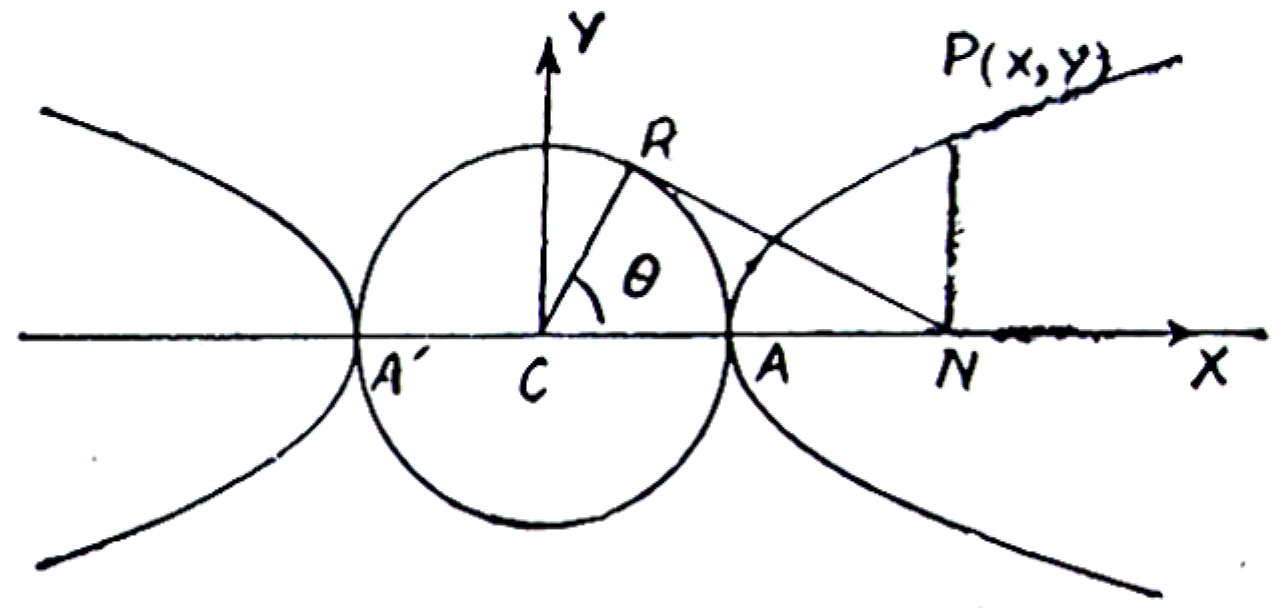
(*i*) $x+2y-3=0$ *w.r.t* the hyperbola
$2x^2-3y^2=1$.

(*ii*) $3x-4y+5=0$ *w.r.t.* the hyperbola
$4x^2-y^2=3$.

3. Find the condition that two points $(x_1, y_1)$, $(x_2, y_2)$ are conjugate *w.r.t.* the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

4. Show that the lines $lx-my=1$ and $l'x+m'y=1$ are conjugate *w.r.t.* the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2ll'-b^2mm'=1$.

**10.6. Parametric Representation.** Find the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.




Let P $(x, y)$ be any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ...(1)$$

From N, the foot of its ordinate, draw NR, the tangent to the auxiliary circle.

Let $\angle NCR=\theta$

Then $CN=CR \sec\theta \quad [\because \sec\theta=\frac{CN}{CR}]$

$=a \sec\theta$

Now substituting this value of $x$ in (1), we have

$$\frac{a^2 \sec^2\theta}{a^2}-\frac{y^2}{b^2}=1$$

or $$\frac{y^2}{b^2}=\sec^2\theta-1=\tan^2\theta$$

$\therefore\ y=b\tan\theta$

Thus $x=a\sec\theta$, $y=b\tan\theta$ are the parametric equations of the hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

$\theta$ being the parameter.

**Cor. 1.** Show that the equation of the tangent at a point $\theta$ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

**Cor. 2.** Show that the equation of the normal at the same point is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$$

**10·61. Asymptotes.** An asymptote is a straight line which meets a curve in two points at infinity but which is not altogether at infinity.

We have seen that the tangent at ($a$ sec $\theta$, $b$ tan $\theta$) to the hyperbola is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

This can be written as

$$\frac{x}{a}-\frac{y}{b}\,[\sin\theta=\cos\theta.$$

If we put $\theta=\frac{\pi}{2}$, the equation of the tangent becomes

$$\frac{x}{a}-\frac{y}{b}=0.$$

This is a line passing through the origin and is a limiting position of the tangent to the curve as the point of contact moves to an indefinite distance along either of the branches.

Similarly if we put $\theta=\frac{3\pi}{2}$

the equation of the tangent becomes

$$\frac{x}{a}+\frac{y}{b}=1$$

which is also a line through the origin and the limiting position of the tangent to the curve as the point of contact moves to an infinite distance along either of the branches. The lines $\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$ or in one equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ are called the **asymptotes** of the hyperbola.

They are tangents that pass through the centre, their points of contact being at an infinite distance.

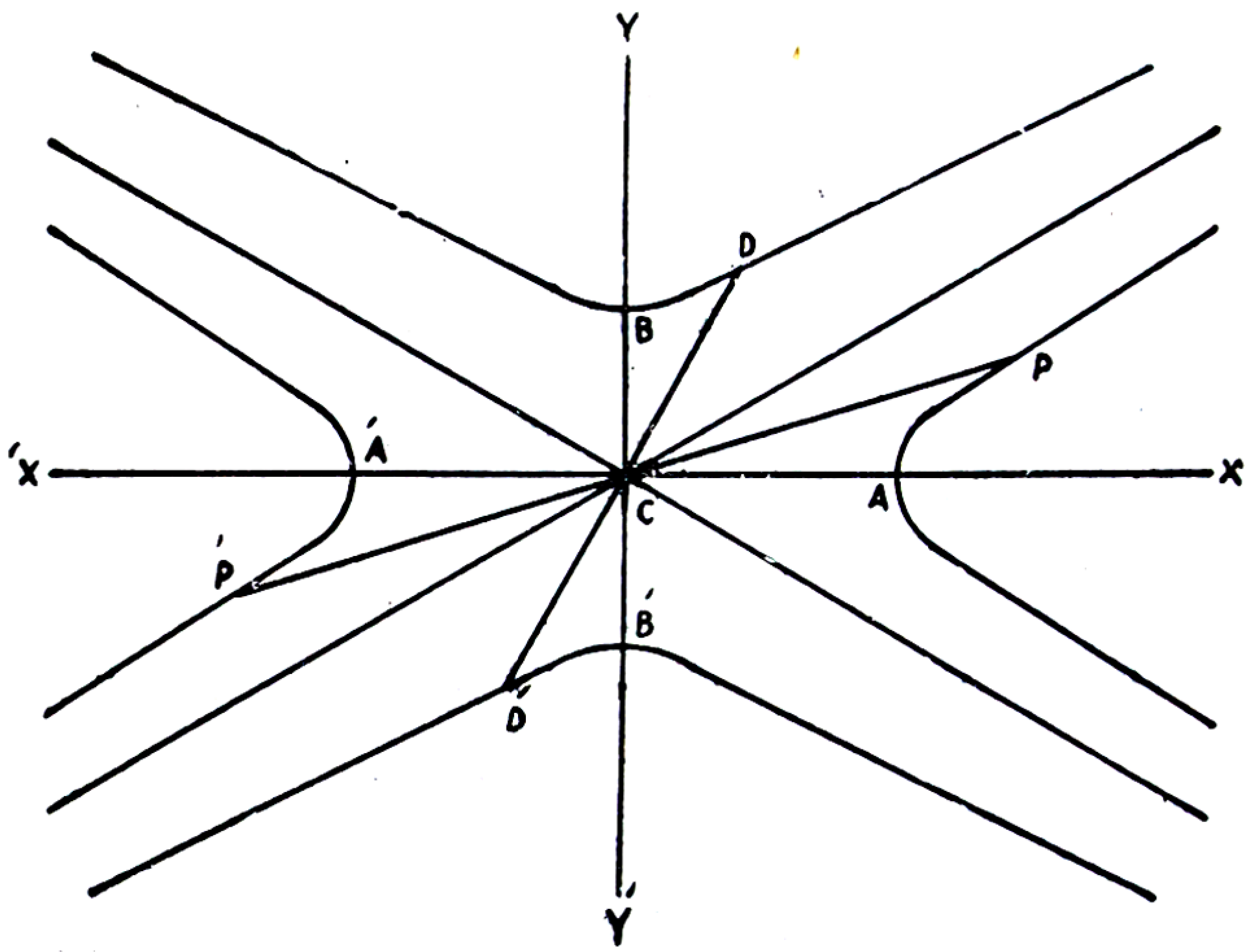
**10·62. Conjugate Hyperbola.** Hyperbolas are said to be conjugate if the transverse axis of one becomes conjugate axis of the other.

Thus the hyperbola

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}==1 \text{ is}$$

conjugate to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

**10·63.** Some properties of conjugate hyperbola.




A'A, B'B are the transverse and conjugate axes of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Now the hyperbola $\dfrac{-x^2}{a^2}+\dfrac{y^2}{b^2}=1$ meets the axis of $y$ in real points $x=0$, $y=\pm b$.

So that BB', A'A are transverse and conjugate axes of the hyperbola $\dfrac{-x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Both curves have the same asymptotes namely the lines $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0$.

The parametric representation for the conjugate hyperbola is $x = a \tan\theta$, $y = b \sec\theta$.

**10·64 Conjugate Diameters.** Two diameters are said to be conjugate when each bisects chords parallel to the other or when one is parellel to the tangents at the ends of the other.

Let a diameter PCP′ of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots(1)$$

meet the curve in points, P, P′. Let P be the point ($a \sec\theta$, $b \tan\theta$).

The equation of the tangent at P is

$$\frac{x}{a} \sec\theta - \frac{y}{b} \tan\theta = 1.$$

The parallel line through the centre is

$$\frac{x}{a} = \frac{y}{b} \sin\theta. \qquad \ldots(2)$$

This line meets the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Where $\frac{y^2}{b^2} = -\sec^2\theta$

which does not give real values for $y$ so that the diameter conjugate to PCP′ does not meet the curve in real points.

But the line (2) meets the conjugate hyperbola in real points where

$$y = \pm b, \sec\theta, \; x = \pm a \tan\theta.$$

We speak of PCP′, DCD′ as conjugate diameters, although P, P′ are points on hyperbola (1) and D, D′ are points on the conjugate hyperbola.

**10·65.** Prove $CP^2 - CD^2 = a^2 - b^2$

Now $CP^2 = a^2 \sec^2\theta + b^2 \tan^2\theta$

$CD^2 = a^2 \tan^2\theta + b^2 \sec^2\theta$

$\therefore \quad CP^2 - CD^2 = a^2 - b^2,$

**10·6.6 The rectangular hyperbola.** When the asymptotes of a hyperbola are at right angles, it is called a rectangular hyperbola.

The lines $\frac{x}{a}-\frac{y}{b}=0,\quad \frac{x}{a}+\frac{y}{b}=0$

are at right angles if $\frac{1}{a^2}-\frac{1}{b^2}=0$

or if $a=b$.

Hence the equation of a rectangular hyperbola is

$$x^2-y^2=a^2.$$

and its asymptotes are $x-y=0$ and $x+y=0$.

**Example 1.** *Show that the eccentricity of a rectangular hyperbola is $\sqrt{2}$.*

The equation of the rectangular hyperbola is $x^2-y^2=1$

$$a=b.$$

Now $b^2=a^2(e^2-1)$

$\therefore\quad 1=e^2-1$ or $e^2=2$.

$\therefore\quad e=\sqrt{2}$.

**Example 2.** *Find the equation to the hyperbola whose eccentricity is $\frac{5}{4}$, whose focus is $(a, 0)$ and directrix is $4x-3y-a=0$.*

Now, SP$=e$PM.

If $(x, y)$ is any point on the hyperbola

$$\sqrt{(x-a)^2+y^2}=\frac{5}{4}\cdot\left(\frac{4x-3y-a}{5}\right)$$

$$\therefore\qquad (x-a)^2+y^2=\frac{(4x-3y-a)^2}{16}$$

or $\quad 7y^2+24ay-24ax-6ay+15a^2=0$

**Example 3.** *Prove that the points*

$$\left[\frac{a}{2}\left(t+\frac{1}{t}\right),\ \frac{b}{2}\left(t-\frac{1}{t}\right)\right] \textit{ lie on the hyperbola.}$$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b_2} = 1$ *for all values of t.*

*and show that the equation of the chord joining two points on the hyperbola whose parameters are* $t_1$ *and* $t_2$ *is*

$$\frac{x}{a}(1+t_1t_2)+\frac{y}{b}(1-t_1t_2)=t_1+t_2$$

*and deduce the equation of the tangent at* $t_1$.

**Sol.** $$\frac{\left[\frac{a}{2}\left(t+\frac{1}{t}\right)\right]^2}{a^2}-\frac{\left[\frac{b}{2}\left(t+\frac{1}{t^2}\right)\right]^2}{b^2}$$

$$=\frac{1}{4}\left[\left(t+\frac{1}{t}\right)^2-\left(t-\frac{1}{t}\right)^2\right]=1,$$

$\therefore$ the point lies on the hyperbola.

The equation of the chord is

$$\left[y-\frac{b}{2}\left(t_1-\frac{1}{t_1}\right)\right]=\frac{\frac{b}{2}\left[t_2-\frac{1}{t_2}-t_1+\frac{1}{t_1}\right]}{\frac{a}{2}\left[t_2+\frac{1}{t_2}-t_1-\frac{1}{t_1}\right]}\left[a-\frac{a}{2}\left(t_1+\frac{1}{t_1}\right)\right]$$

or $$\frac{x}{a}(1+t_1t_2)+\frac{y}{b}(1-t_1t_2)=t_1+t_2$$

Hence the equation of the tangent at the point $t_1$

$$\frac{x}{a}(1+t_1^2)+\frac{y}{b}(1-t_1^2)=2t_1$$

**Example 4.** *Find the locus of the poles of Normal chords of the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

If $x_1y_1$ be the pole of the variable normal

$ax \sin \phi+by=(a^2+b^2) \tan \phi$ ...(1)

it must be the as $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$ ...(2)

Comparing (1) and (2)

$$\frac{a^3}{x_1}\sin\phi=\frac{-b^3}{y}=(a^2+b^2)\tan\phi$$

Eliminating $\phi$ *i.e.*, $\sin\phi=\dfrac{-b^2x_1}{a^3y_1}$

$$\cos\phi=\frac{(a^2+b^2)x_1}{a^3}$$

we get

$$\left(\frac{b^3x_1}{a^3y_1}\right)^2+\left(\frac{a^2+b^2}{a^3}\right)^2x_1^2=1$$

or $a^6y^2-b^6x^2=(a^2+b^2)^2x^2y^2$ as the required locus.

## Exercises X (E)

1. Prove that the $lx+my=n$ touches the rectangular hyperbola $xy=e^2$ if $n^2=4lmc^2$.

2. Show that tangents to a rectangular hyperbola at the extremities of its latera recta pass through the vertices of the conjugate hyperbola.

3. If the polars of $(x_1, y_1)$, $(x_2, y_2)$ with respect to $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ are at right angles, then $\dfrac{x_1x_2}{y_1y_2}+\dfrac{a^4}{b^4}=0$.

4. The poles with respect to $y^2-4ax=0$ of tangents to $x^2+y^2-a^2=0$ are on the hyperbola $4x^2-y^2=4a^2$.

5. Find the equation of an hyperbola when referred to its asymptotes as axes of co-ordinates.

6. If the normal at four points on a hyperbola meet in a point, prove that the sum of the eccentric angles of their feet is equal to an odd multiple of $\pi$.

7. Find the equation to the hyperbola whose asymptotes are given by $x+2y+3=0$ and $3x+4y+5=0$ and which pass through the point $(1, -1)$.

**Sol.** The hyperbola is given by

$$(x+2y+3)(3x+4y+5)=c$$

where $c$ is determined by the fact that the point $(1, -1)$ lies on the hyperbola.

## ANSWERS

### Exercises I (B) (Page 4)

**1.** (*i*) 5. (*ii*) $\sqrt{13}$ (*iii*) $\sqrt{37}$ (*iv*) $\sqrt{2(a^2+b^2)}$.
(*v*) $\sqrt{a^2+c^2+2b^2-2bc-2ab}$. (*vi*) $a\sqrt{2}\sqrt{1-\cos(\alpha-\beta)}$.

### Exercises I (C) (Page 10)

**1.** (*i*) (3, 4). (*ii*) (−3, 0). (*iii*) ($a$, $c$).

**2.** (*i*) $\frac{\sqrt{298}}{2}$, $\sqrt{51}$, $\frac{\sqrt{130}}{2}$. (*ii*) $(-x_1-y_1)$.
(*iii*) (−5, 3), (1, −3)(7, 5).

**3.** (*i*) $\left(\frac{11}{5}, \frac{16}{5}\right)$. (*ii*) $\left(\frac{9}{2}, -3\right)$. (*iii*) $\left(\frac{-7}{4}, 2\right)$
(*iv*) (11, −12). (*v*) $\left(\frac{-25}{6}, \frac{-33}{6}\right)$.

. (*i*) (7 : −7). (*b*) (5 : 1). (1 : 6).

**5.** $(\frac{8}{3}, \frac{1}{3})$. **6.** $(-25, -17\frac{1}{2})$; $(-15, -15)$, $(-5, -12\frac{1}{2})$.

**7.** (*i*) (2, −1). (*ii*) (2, 3). (*iii*) $\frac{10}{3}, \frac{1}{3})$.

**8.** $(\frac{180}{77}, \frac{915}{77})$ (*ii*) $\left[\frac{\sqrt{125}-2\sqrt{20}+3\sqrt{45}}{\sqrt{20}+\sqrt{45}+\sqrt{125}}\right.$.
$\left.\frac{-2\sqrt{125}+\sqrt{20}(4)-6\sqrt{45}}{\sqrt{125}+\sqrt{20}+\sqrt{45}}\right]$.

### Exercises I (D) (Page 17)

**1.** (*i*) (+8). (*ii*) (26). (*iii*) (40). (*iv*) $(\frac{11}{2})$. (*v*) (2).
(*vi*) $a^2[t_1{}^2t^2-t_1t_2{}^2+t_2{}^2t_3-t_3{}^2t_2+t_1t_3{}^2[$.

## Exercises II (Page 17)

**1.** $x^2+y^2-6x-4y-12=0$. **2.** (*i*) $y=2x$ (*ii*) $3y=4x$.
**3.** $x^2-4x-6y+13=0$. **4.** $4x^2+4y^2+3x-27y+36=0$.
**5.** (*i*) $x^2+y^2-4y-2x+4=0$. (*ii*) $12x-7=0$.
**6.** $(2, 1)$; $(\sqrt{5}, 0)$ ; $(-2, -1)$. **11.** 0.
**12.** $(\pm 3, 0)$ ; $(0, \pm 3)$.

## Exercises III (A) (Page 20)

**3.** (*i*) $-7$ ; $\frac{1}{2}$ ; $\frac{-2}{11}$. (*ii*) $-7$ ; $\frac{4}{3}$ ; $\frac{-3}{4}$.

**5.** $-7$ , $\frac{1}{6}$ ; $\frac{-8}{11}$ ; $\frac{-1}{2}$ ; $\frac{-6}{7}$ ; $\frac{-9}{5}$.

## Exercises III (B) (Page 24)

**1.** (*i*) $y=7$. (*ii*) $x+7=0$. **2.** $y=7$, $y=11$ and $y=16$.
**3.** (*i*) $y+x=4$. (*ii*) $y+\sqrt{3}x+6=0$. (*iii*) $y=x+1$.
**4.** (*i*) $6x-5y=30$. (*ii*) $x+y=15$. (*iii*) $x-y=4$.
**5.** $m=\frac{1}{5}$, $c=6\frac{2}{5}$. **6.** $3x+2y-12=0$.

## Exercises III (C) (Page 29)

**1.** (*i*) $y=\frac{2}{3}x+\frac{2}{3}$. (*ii*) $y=-\frac{4}{5}x+\frac{1}{5}$.

(*iii*) $y=-x\cot\alpha+p\operatorname{cosec}\alpha$. (*iv*) $y=-\frac{b}{a}x+b$.

**2.** (*i*) $\frac{x}{-8}+\frac{y}{6}=1$. (*ii*) $x+\frac{y}{\frac{-5}{3}}=1$.

(*iii*) $\frac{x}{\frac{-c}{m}}+\frac{y}{c}=1$, (*iv*) $\frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1$.

**3.** (*i*) $\frac{3}{5}x+\frac{4}{5}y=1$. (*ii*) $\frac{-1}{2}x-\frac{\sqrt{3}}{2}y=\frac{7}{2}$.

(*iii*) $\frac{-m}{\sqrt{1+m^2}}x+\frac{y}{\sqrt{1+m^2}}=\frac{c}{\sqrt{1+m^2}}$.

(*iv*) $\frac{-12}{13}x+\frac{5}{13}y=\frac{9}{13}$. **5.** $3:-2$. **6.** $3:4$.

## Exercises III (D) (Page 34)

**1.** $y+x=0.$ **2.** $10x+7y-11=0.$ **3.** $4x-5y+13=0.$

**4.** $2x-y(t_1+t_2)+2at_1t_2=0.$

**5.** $x\cos\dfrac{\theta+\phi}{2}+y\sin\dfrac{\theta+\phi}{2}=a\cos\dfrac{\theta-\phi}{2}.$

**6.** $\dfrac{x}{a}\cos\dfrac{\theta+\phi}{2}+\dfrac{y}{b}\sin\dfrac{\theta+\phi}{2}=\cos\dfrac{\theta-\phi}{2}.$

**7.** $7x+y-11=0,\ x+3y+7=0,\ 3x-y+1=0.$

**10.** $1:1.$ **11.** $(6, 8).$ **12.** $-\frac{1}{4},\ \frac{1}{3}.$

**14.** $x+y=5.$ **15.** $\dfrac{x}{a}+\dfrac{y}{b}=2.$

**16.** $4x-3y=0,\quad \dfrac{x}{6}+\dfrac{y}{8}=1.$ **17.** $-3\sqrt{2}.$

## Exercise IV (A) (Page 40)

**1.** (*i*) $60°$ (*ii*) $45°$ (*iii*) $(\alpha-\beta)$.

**2.** $\tan^{-1}\dfrac{ab'-a'b}{aa'+bb'}$

**4.** (*i*) $3x-4y+6=0$ (*ii*) $x+y-2=0$

(*iii*) $a(x-x_1)+b(y-y_1)=0$

**5.** (*i*) $12x-5y=0$

(*ii*) $B(x-x_1)-A(y-y_1)=0$

**6.** $xx'-yy'=x'^2-y'^2$

**7.** (*i*) $mx-ly+lk-mk=0$

(*ii*) $qx+py=aq+bp$

**8.** $x-3y-4=0.$ 9. $x+2y-9=0$ ; $3x-5y+8=0$ ;

$6x+y-19=0$ 10. (*i*) $3x+y-17=0$ ; $x-3y+11=0$

(*ii*) $y-k=\dfrac{m+\tan\alpha}{1-m\tan\alpha}(x-h)$ ; $y-k=\dfrac{m-\tan\alpha}{1+m\tan\alpha}(x-h)$

## Exercises IV (B) (Page 46)

**1.** (*i*) 1, 2 (*ii*) (3, 5) (*ii*) $\frac{ab}{a+b}, \frac{ab}{a+b}$

**2.** (*i*) 36 (*ii*) $c^2$ **3.** (*i*) $4x+5y=0$

(*ii*) $64x-23y=59$ (*iii*) $6x+13y=25$

(*iv*) $\frac{ax+by+c}{ax'+by'+c}=\frac{a'x+b'y+c'}{a'x'+b'y'+c'}$

(*v*) $62x+93y-165=0$

**4.** $3x-2y-7=0$ ; $3x+y-2=0$ ; $x-4y-3=0$,

$\left(\frac{11}{13}, \frac{-7}{13}\right)$ **6.** $\Sigma\ m_1\ (c_2-c_3)=0$

**7.** $a=5$

## Exercises IV (C) (Page 51)

**1.** (*i*) 1 (*ii*) 3 (*iii*) $\frac{b^2-a^2}{\sqrt{b^2+a^2}}$

(*iv*) $\frac{a^2+ab-b^2}{\sqrt{a^2+b^2}}$ (*v*) $\sqrt{h^2+k^2}$

**3.** (*i*) $1\frac{1}{5}$ (*ii*) $(\frac{61}{25}, \frac{127}{25})$ (*iii*) $4x+3y=25$

**4.** $(\frac{4}{5}, \frac{7}{5})$ ; $(\frac{1}{5}, \frac{7}{5})$ ; $\frac{2}{5}$.

**5.** $\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{13}}, \frac{5}{\sqrt{13}}$ **6.** (*i*) $\frac{24}{5}$ (*ii*) $\frac{7}{2}$

(*iii*) $\frac{c'-c}{\sqrt{a^2+b^2}}$ (*iv*) $\frac{c-d}{\sqrt{1+m^2}}$

**7.** $\left(\frac{26}{7}, \frac{26}{7}\right)$ ; $\left(\frac{-24}{7}, \frac{-24}{7}\right)$ **8.** $3x+4y+10=0$

and $3x+4y-30=0$ **9.** $(2, -3)$

**10.** $\frac{14}{\sqrt{13}}$ $\frac{28}{\sqrt{10}}$ $\frac{28}{\sqrt{34}}$

## Exercises IV (D) (Page 55)

**1.** $(3, -4)$ lies on the same side as origin ; $(4, 2)$ on the opposite side.

**4.** (*i*) $3x-11y=0$ and $99x+27y-130=0$
(*ii*) $x(\cos\alpha\pm\cos\beta)+y(\sin\alpha\pm\sin\beta)-(p\pm p')=0$

**5.** (*i*) $14x+8y-271=0$ (*ii*) $x-3y-5=0$
(*iii*) $x+y=0$ **6.** (*i*) $7x-56y+228=0$
(*ii*) $11x-3y+1=0$ (*iii*) $209x+437y-217=0$

**7.** $(0, 1)$

## Revision Exercises I. (Page 56)

**6.** $(2, -1)$. **8.** $3y^2-x^2=0$. **9.** $2x-y=2$.
**10.** $(0, 0)$, $(\frac{64}{17}, -\frac{16}{17})$. **13.** 2.
**18.** $x-y+1=0$, $7x-4y+1=0$, $8x-5y-1=0$.
**19.** $45°$. **20.** $4x-y-5=0$, $x+4y-14=0$.
**21.** 2, 2, 2. **22.** $x+7y+(6\pm5\sqrt{2})=0$.
**23.** $7x+26y-33=0$. **24.** $c^2$.
**27.** $x+y=4$. **28.** $x(x-h)+y(y-k)=0$.
**29.** $(\frac{59}{16}, -\frac{25}{8})$. **30.** $(-4, -4)$.

## Exercise V (A) (Page 64)

**1.** (*i*) $x^2+y^2-6x+4y-3=0$ (*ii*) $4x^2+4y^2-8ax-8ay+7a^2=0$
(*iii*) $x^2+y^2-6x+10y-19=0$

**2.** (*i*) $(0, 0)$, 7 (*ii*) $(4, 3)$, 7 (*iii*) $(3, -4)$, 4
(*iv*) $\left(\frac{a}{2}, \frac{b}{2}\right)$; $\frac{\sqrt{a^2+b^2}}{2}$ (*v*) $(-1, 0)$ ; 0
(*vi*) $(\frac{5}{6}, 1)$ ; $\sqrt{\frac{13}{61}}$ (*vii*) $\left(\frac{-2}{5}, \frac{4}{5}\right)$ ; 2
(*viii*) $\left(\frac{3a}{2}, \frac{3a}{4}\right)$ ; $\frac{7a}{4}$

**3.** $x^2+y^2-4x-6y+4=0$.
**4.** $x^2+y^2-8x-10y+1=0$. **5.** $x^2+y^2-4x-6y-87=0$.
**6.** (*i*) $x^2+y^2-4x-6y-3=0$.
(*ii*) $x^2+y^2-8x-10y-8=0$.
**7.** $(10, 7)$

**8.** (*i*) $3x^2+3y^2-5x-5y=0$
(*ii*) $x^2+y^2+4x-3y=0$
(*iii*) $x^2+y^2-13x-13y+52=0$
(*iv*) $x^2+y^2-4x-4y+7=0$
(*v*) $x^2+y^2+x-5y-2=0$
(*vi*) $x^2+y^2-13x-5y+16=0.$

**9.** $x^2+y^2-ax-by=0$ ; $\left(\frac{a}{2}, \frac{b}{2}\right)$ ; $\frac{\sqrt{a^2+b^2}}{2}$

**10.** $5x^2+5y^2-8x+4y-5=0$

**11.** (*i*) $x^2+y^2-5x+3y-22=0$
(*ii*) $x^2+y^2-10x+15=0$
(*iii*) $x^2+y^2+a(\sin\theta-\cos\theta)x$
$-b(\sin\theta+\cos\theta)\,y-(a^2-b^2)\sin\theta\cos\theta=0.$

**12.** (*i*) $x^2+y^2-6x-2y+5=0$
(*ii*) $x^2+y^2-4x-2y-5=0$
(*iii*) $a(x^2+y^2)-(a^2-b^2)x-b^2a=0$

**13.** $(x-2)^2+(y+1)^2=16$ and $(x-6)^2+(y-3)^2=16.$

**14.** (*i*) $x^2+y^2-10x-10y+25=0$
(*ii*) $x^2+y^2-2ax-2ay+a^2=0$
(*iii*) $x^2+y^2+2(5\pm\sqrt{12}(x+y)+37+10\sqrt{12}=0$
(*iv*) $x^2+y^2-6x+4y+9=0$

**15.** $x^2+y^2-3x+7y+2=0$ and $x^2+y^2-3x+y+2=0$

**16.** $x^2+y^2-5x+5y=0.$

**17.** $x^2+y^2-13x+\frac{13+12\sqrt{5}}{2}y+36+0.$

## Exercises V (B) (Page 72)

**1.** $2x+3y=13$ ; $3x-2y=0$ **2.** $3x+4y=26$ ; $4x-3y-18=0.$

**3.** $3x+7y=93$ and $3x-7y=64$
Normal $7x-3g=43$ and $7x+3y=55.$

**4.** $5x+12y\pm39=0.$ **5.** $4x+3y\pm5\sqrt{85}=0.$

**7.** $5x+12y-32=0$ and $5x-12y-152=0.$

**8.** No. **9.** $a\cos^2\alpha+b\sin^2\alpha\pm\sqrt{a^2+b^2\sin^2\alpha}$

**10.** (*i*) $3x-4y=0$ ; $3x-4y+10=0$
(*ii*) $4x+3y=15$ ; $4x+3y=5.$

### Exercises V (C) (Page 77)

**1.** $\left(\pm\frac{2}{\sqrt{13}}, \pm\frac{3}{\sqrt{13}}\right)$ **2.** $(-1, -1)$ ; $\left(\frac{1}{5}, \frac{7}{5}\right)$ ; $\frac{6}{\sqrt{5}}$ **3.** $\left(\frac{-12}{5}, \frac{-16}{5}\right)$

**4.** $(mh-k+c)^2=a^2(1+m^2)$ **5.** $2x+y\pm10\sqrt{5}=0.$

**6.** $x+y\pm3\sqrt{2}=0.$

**7.** $a^2l^2+a^2m^2=n^2$ ; $\left(\frac{-a^2l}{n}, \frac{-a^2m}{n}\right)$

**8.** $p=\pm a$ ; $(p\cos\alpha, p\sin\alpha)$ **9.** $\frac{5}{4}$ ; $\frac{-45}{4}$ **10.** (*i*) $g=\pm\sqrt{c}$ (*ii*) $f=\pm\sqrt{c}.$ **12.** $y=\sqrt{3}x\pm6.$

**13.** $(-1, -1)$ **14.** $y=\pm5$, $y=\pm\sqrt{3}x\pm10.$

**15.** $4x^2+4y^2-4cx\pm4cy+c^2=0.$

### Exercises V (D) (Page 82)

**1.** (*i*) $3x\pm4y+15=0$ (*ii*) $x=4$ ; $3x-4y+8=0.$

**2.** $3x-2y-13=0.$ **3.** (*i*) $2xy=k_1(x^2-a^2)$ (*ii*) $y^2-a^2=k_2(x^2-a^2)$ (*iii*) $k_3(y^2-a^2)=2xy.$

### Exercises V (E) (Page 86)

**1.** (*i*) $2x-5y+10=0$ (*ii*) $2x-7y-9=0.$

**2.** (*i*) $(2, 1)$ (*ii*) $\left(\frac{a^2\cos\alpha}{p}, \frac{a^2\sin\alpha}{p}\right)$

**3.** $y^2+2ax+a^2=0$

### Exercises VI (A) (Page 92)

**1.** $\sqrt{\left|\frac{1}{2}(a-b)^2-4c\right|}$ **2.** $x^2+y^2+4x-7y+5=0$

**3.** $5(x^2+y^2)+36x-116y-140=0.$

**7.** $x^2+y^2-x-2y=0$ ; $x^2+y^2+4x-2y+4=0.$

### Exercises VI (B) (Page 97)

**1.** 3 **3.** $x^2+y^2+6x-3y=0.$

**4.** $x^2+y^2+7x-9y-6=0.$ **5.** $x+3y+7=0.$

**6.** (*i*) $26x-23y-292=0$ (*ii*) $x-y=0$ (*iii*) $5x-3y+5=0$.
**7.** (*i*) $(2, 1)$ (*ii*) radical centre is at infinity.
**8.** $x^2+y^2+2x+2y+1=0$.

## Revision Exercises II (Page 98)

**1.** $b(x^2+y^2)-(b^2-a^2)y-a^2b=0$.
**4.** $\left[\frac{lmf-m^2g+l}{l^2+m^2} \cdot \frac{lmg-l^2f+m}{l^2+m^2}\right]$
**7.** $(x+\frac{1}{2})^2+(y+\frac{3}{2})^2=8$. **8.** Circle.
**18.** 24 or $-56$. **19.** $(-1, 1)$.
**21.** The vertices of the triangle are $(1, 1)$, $(-2, 6)$, $(\frac{3}{2}, \frac{5}{2})$.

## Exercises VII (A) (Page 103)

**2.** (*i*) $\frac{1}{3}$. (*ii*) $\frac{1}{4}$. **3.** $(0, 0)$ ; $(4a, 4a)$. **4.** $\pm 2a$.

## Exercises VII (B) (Page 105)

**2.** $2x+y=0$ ; $(8, -2)$ ; $1\sqrt{5}$ ; $x-2y+5=0$
**4.** (*a*) $(-1, 2)$ ; $(0, 2)$ ; $y=2$, $x+2=0$.
(*b*) $(1, -1)$ ; $(1, 2)$, $x=1$ ; $y+4=0$.

## Exercises VII (C) (Page 119)

**1.** (*i*) $(a, 2a)$ $\left[\frac{a}{9}, \frac{-}{3}\right]$. (*ii*) $(3, 2)$, $\left[\frac{1}{3}, \frac{-2}{3}\right]$.
(*iii*) $(-3, -3)$. **2.** $4\sqrt{2}$.
**3.** (*a*) $y-x=a$, $x+y+a=0$ ; $x+y=3a$, $x-y=3a$.
(*b*) $3x-4y+12=0$ ; $4x+3y-34=0$
(*c*) $x-4y+24=0$ ; $4x+y-108=0$. **4.** $64x+16y+7=0$
**5.** $x+4y+20=0$ ; $(20, -10)$. **6.** $(\frac{9}{2}-3)$. **7.** (*i*) $x-y+3=0$
(*ii*) $3x-y\sqrt{3}+a=0$.

**10.** $\{at_1t_2, a(t_1+t_2)\}$. **13.** (*i*) $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$ ;
(*ii*) $x=0$. **17.** (*a*) $4x+3y+1=0$ ; (*b*) $x=0$ ;
(*c*) $y^2=a(x-a)$. **18.** $y^2=2a(x-a)$. **19.** $y^2=2ax$.
**20.** $y^2-ky=2a(x-h)$.

## Exercises VII (D) (Page 130)

**1.** (*b*) $x+4y+6=0$, $x-8y+24=0$.
**3.** (*a*) $y=bx$. (*b*) $x=a/c$. **4.** $y=ad$. **5.** $y=(x-a)\tan 2d$.

**6.** $y^2=gx^2+2ax$. **7.** (*i*) $y^2-4ax=(x+a^2)$.
(*ii*) $y^2-3x^2-10ax-3a^2=0$.

**8.** (*a*) Only one real normal $x+y=3a$.
(*b*) $x-y-3a=0$ (coincident) $2x+y=12a$,

**9.** $y^2=a(x-3a)$. **10.** $y^2=4ax$. **11.** $y=(x-a)\tan k$.

## Exercises VII (E) (Page 134)

**1.** $3x-y+6=0$. **3.** (*i*) $(-\frac{5}{3}, -\frac{8}{3})$. (*ii*) $(3a, 4a)$.
**6.** $(1, 2)$ ; $(4, 4)$.

## Exercises VII (F) (Page 136)

**1.** Latus rectum is $x=\dfrac{B^2-A^2-C}{2A}$, vertex is $\left[\dfrac{B^2-C}{2A} -B\right]$

**4.** $[a(t^2+t_1t_2+t_2^2)+2a^2, -at_1t_2(t_1+t_3)]$.

## Exercises VIII (A) (Page 145)

**1.** (*i*) $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$. (*ii*) $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$.

(*iii*) $\dfrac{x^2}{50}+\dfrac{y^2}{25}=1$. (*iv*) $\dfrac{x^2}{256}+\dfrac{y^2}{240}=1$.

(*v*) $\dfrac{x^2}{81}+\dfrac{y^2}{72}=1$. (*vi*) $\dfrac{x^2}{1}+\dfrac{9y^2}{4}=1$.

(*vii*) $\dfrac{x^2}{225}+\dfrac{y^2}{144}=1$.

**2.** (*i*) $\dfrac{\sqrt{3}}{2}$, (*ii*) $\sqrt{\dfrac{2}{3}}$, (*iii*) $\dfrac{1}{\sqrt{2}}$.

**4.** (*i*) $\sqrt{\frac{1}{5}}, \frac{4}{5}, (0, \pm\dfrac{1}{\sqrt{10}})$, (*ii*) $\sqrt{\frac{2}{3}}, \dfrac{2a}{3}, (\pm a\sqrt{\frac{2}{3}}, 0)$.

(*iii*) $\dfrac{\sqrt{7}}{4}, \frac{9}{2}, (\pm\sqrt{7}, 0)$, (*iv*) $\dfrac{\sqrt{5}}{3}, \frac{8}{3}, (9, \pm\sqrt{5})$.

(*v*) $\frac{1}{2}$ eccentricity $3\sqrt{2}$, $(0, +\sqrt{2})$.

**5.** (*a*) $\frac{1}{2}$, 2. (*b*) $\left[\pm\dfrac{1}{\sqrt{6}}, 0\right]$, $\left[\dfrac{1}{\sqrt{6}}, \pm\dfrac{\sqrt{2}}{3}\right]$,
$\left[-\dfrac{1}{\sqrt{6}}, \pm\dfrac{\sqrt{2}}{3}\right]$.

7. Centre is $(0, 3)$, $S'$, $(-1, 4)$; equation to the directrix is $x-y+11=0$ and to the ellipse, $6x^2+8y^2=48$.

10. Zero $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$. 11. $60°$. 12. $\frac{\pi}{4}, \frac{5\pi}{4}$.

13. $\frac{\pi}{4}$ if $a^2=6, b^2=2$. 14. (a) $\left[\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right]$.

(b) $\left[\frac{a}{2}, \frac{\sqrt{3}b}{2}\right]$.

## Exercises VIII (B) (Page 156)

1. (a) 4. (b) $\sqrt{2}$. 2. $(1, \frac{3}{2})$. 3. $y=3x+\sqrt{\frac{155}{12}}$.
4. $4x+3y=\sqrt{\frac{7}{3}}$. 5. $y=\pm x\pm\sqrt{a^2+b^2}$.
6. $4x+2\sqrt{3}y=8\sqrt{3}$, $2\sqrt{3}x-4y+2=0$.

16. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 17. $3x+4y=7$. 18. $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

19. $\frac{x^2}{a^2(1+e^2)^2}+\frac{y^2}{b^2}=\frac{1}{4}$. 20. $(a^2+b^2)(b^2x^2+a^2y^2)=a^2b^2=0$ $(b^4x^2+a^4y^2)$.

21. (a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{x}{a}=0$. (b) $b^2x(x-h+a^2y)(y-k)=0$.

## Exercises VIII (C) (Page 165)

1. $5y=4x+25$, $4y=x+13$. 2. $y=x+3$, $5y=-x+9$.
4. $x^2-2xy\cot 2\alpha-y^2=a^2-b^2$. 5. $cx^2-2xy=ca^2$.
6. $d^2(x^2-a^2)^2=4(b^2x^2+a^2y^2-a^2b^2)$.
7. $k(x^2-a^2)^2=2(x^2y^2+b^2x^2+a^2y^2-a^2b^2)$.
8. $gy^2-2xy=gb^2$.

## Exercises VIII (D) (Page 167)

1. (i) $4x+3y=1$. (ii) $16y-9x=5$.

2. (i) $(\frac{1}{6}, \frac{2}{9})$. (ii) $\left[\frac{-9}{20}, \frac{12}{5}\right]$.

3. $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=7$.

## Revision Exercises No. IV (Page 171)

1. (*i*) $x^2/36+y^2/16=1$. (*ii*) $x^2/36+y^2/18=1$.
(*iii*) $x^2+4y^2=1$.

2. (*a*) (*i*) $x^2(1-e^2\cos^2\alpha)+y^2(1-e^2\sin^2\alpha)-2e^2\cos\alpha\sin\alpha.\,x$
$2pe^2\sin\alpha.\,y-a^2e^2=0$.
(*ii*) $7x^2+7y^2+2xy+10x-10y+7=0$.
(*b*) $x+y=0$ ; $(-\frac{5}{6}, \frac{5}{6})$.

3. (*i*) $e=2\sqrt{2}/3$ ; foci $(\pm 6\sqrt{2}, 0)$ ; directrices ; $x=\pm 27/\sqrt{2}$ ; L. R.$=2$. Equations of L. R. $x=\pm 6\sqrt{2}$.

(*ii*) $e=\sqrt{15}/4$ ; foci $(0\pm 5\sqrt{15}/4)$, directrices : $y=\pm 20/\sqrt{15}$ ; L. R.$=5/8$. Equations of L. R. $y=\pm 5\sqrt{15}/4$.

(*iii*) $e=\sqrt{7}/4$ ; foci $(0, \pm\sqrt{7})$ ; directrices : $y=\pm 16/\sqrt{5}$. L. R.$=\frac{9}{2}$. Equations of L. R. $y=\pm\sqrt{7}$.

8. Tangent $ex\pm y=a$ ; Normal : $xa\mp aey+e(b^2-a^2)=0$.
Tangent : $ex\mp y+a=0$, Normal : $ax\pm a(y\mp b^2-{}^2a)e=0$.

9. $64x\pm 27\sqrt{3}y=2881$. 10. $x^2+y^2=a^2+b^2$.

12. (*i*) $\left[\frac{\pm a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right]$, (*ii*) $\left[\frac{\pm a^2}{\sqrt{a^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2+b^2}}\right]$.

14. $$\left\{\frac{a^2-b^2}{a}\cos\theta_1\cos\theta_2\,\frac{\sin\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_2-\theta_1}{2}}\right.$$

$$\left.\frac{a^2-b^2}{b}\sin\theta_1\sin\theta_2\,\frac{\sin\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}\right\}$$

17. $bx=ay\cot\alpha$.

22. (*i*) $ax\pm by=\sqrt{a^4+a^2b^2+b^4}$, (*ii*) $y=x+\sqrt{a^2+b^2}$.

## Exercises X (A) (Page 182)

1. $65x^2-36y^2=441$. 2. $x^2-y^2=8$. 3. $4x^2-y^2=4$.

**4.** $6\ ;\ 4\ ;\ (\pm\sqrt{13},\ 0)\ ;\ \sqrt{\tfrac{13}{3}}\ ;\ 2\tfrac{2}{8}$. **5.** $2\ ;\ \left(\pm\dfrac{4}{\sqrt{13}}, 0\right)$.

**6.** $(0\pm\sqrt{29})\ ;\ y=\pm\dfrac{25}{\sqrt{29}}\ ;\ \dfrac{\sqrt{29}}{5}\ ;\ \tfrac{8}{5}$.

## Exercise X (B) (Page 186)

**1.** Tangents $2x-y=4$ and $2x+y+4=0$
Normals $x+2y-7=0$ and $x-2y+7=0$.

**2.** $(4, 2)\ ;\ (-4, -2)\ ;\ 4\sqrt{5}$. **3.** $y=3x\pm\sqrt{\tfrac{115}{12}}$.

**4.** $3y+4x\pm1=0$.

ASL-259 Intermediate Coordinate Geometry
Prem Singh, V Prakash, Jan Mohammad,
Amrik Singh -
(211 Pages) Gian Chand & Co Ambala Cantt.
ASL-260 Enduring Passion
Dr. Marie Stopes (1953) 163 Pages
ASL-261 Wonders of the Past & Present
C.A. Parkhurst - 1949
Macmillan & Company Ltd. London
(151 Pages)

15/01/10

Overdue charges will be levied under rules for each day if the book is kept beyond the date stamed above.

ASL-259 Intermediate Coordinate Geometry
Prem Singh, V Prakash, Jan Mohammed,
Amrik Singh.
(211 Pages) Gian Chand & Co Ambala Cantt.

ASL-260 Enduring Passion
Dr. Marie Stopes ( 1953 ) 163 pages

ASL-261 Wonders of the Past & Present
C.A. Parkhurst. 1949
Macmillan & Company Ltd. London
(151 Pages)